Ground states for a linearly coupled indefinite Schrödinger system with steep potential well

Kuan-Hsiang Wang

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Abstract

- We are concerned with the investigation of a class of linearly coupled Schrödinger systems with steep potential well, which arises in nonlinear optics.
- The existence of positive ground states is investigated by exploiting the relation between the Nehari manifold and fibering maps.
- This is a joint work with Prof. Tsung-fang Wu and Prof. Ying-Chieh Lin.

Systems of linearly coupled Schrödinger equations

The propagation of optical pulses in a nonlinear dual-core coupler can be described in terms of two linearly coupled Schrödinger equations:

$$\begin{cases} -i\frac{\partial\Psi}{\partial t} = \Delta\Psi - a(x)\Psi + |\Psi|^2\Psi + \beta\Phi, & x \in \mathbb{R}^N,\ t \geq 0, \\ -i\frac{\partial\Phi}{\partial t} = \Delta\Phi - b(x)\Phi + |\Phi|^2\Phi + \beta\Psi, & x \in \mathbb{R}^N,\ t \geq 0, \end{cases} \tag{1}$$

where

- Ψ and Φ are the complex valued envelope functions,
- a and b are potential functions, and
- β, which is the normalized coupling coefficient between the two cores, is equal to the linear coupling coefficient times the dispersion length.

If we consider a standing wave solution (soliton) for system (1) of the form

$$(\Psi(t,x), \Phi(t,x)) = \left(e^{-i\omega t} u(x), e^{-i\omega t} v(x)\right),$$

where u, v are real functions decreasing to zero at infinity and $\omega > 0$ is a parameter. Then (u, v) solves the following system

$$\begin{cases} -\Delta u + (a(x) - \omega) u = u^3 + \beta v, & x \in \mathbb{R}^N, \\ -\Delta v + (b(x) - \omega) v = v^3 + \beta u, & x \in \mathbb{R}^N. \end{cases} \tag{2}$$

Systems of linearly coupled Schrödinger equations

We are concerned with the following system of linearly coupled Schrödinger equations:

$$\begin{cases} -\Delta u + V_1(x)\, u = f(u) + \beta(x) v, & x \in \mathbb{R}^N, \\ -\Delta v + V_2(x)\, v = g(v) + \beta(x) u, & x \in \mathbb{R}^N, \\ (u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N), \end{cases} \tag{3}$$

where

- the potentials $V_1(x) \not\equiv 0$, $V_2(x) \not\equiv 0$ are continuous and nonnegative,
- the nonlinear terms f, g are continous, and
- the coupling function $\beta(x) \not\equiv 0$ is continuous and nonnegative.

Weak solutions

Definition 1

We say that a pair of functions (u, v) is a (weak) solution of system (3) if

$$\int_{\mathbb{R}^N} \nabla u \nabla \phi + V_1(x) u \phi dx = \int_{\mathbb{R}^N} f(u) \phi dx + \int_{\mathbb{R}^N} \beta(x) v \phi dx,$$
$$\int_{\mathbb{R}^N} \nabla v \nabla \psi + V_2(x) v \psi dx = \int_{\mathbb{R}^N} g(v) \psi dx + \int_{\mathbb{R}^N} \beta(x) u \psi dx,$$

for all $\phi, \psi \in C_0^\infty(\mathbb{R}^N)$. Moreover, we call a solution of system (3) is nontrivial if $(u, v) \neq (0, 0)$, is nonnegative if $u, v \geq 0$, and is positive if $u, v > 0$.

Variational setting

Associated with system (3), we can define the energy functional

$$I(u,v)=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2+V_1(x)u^2dx+\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2+V_2(x)v^2dx$$
$$-\int_{\mathbb{R}^N}F(u)dx-\int_{\mathbb{R}^N}G(v)dx-\int_{\mathbb{R}^N}\beta(x)uvdx,$$

where $F(u)=\int_0^u f(s)ds$ and $G(v)=\int_0^v g(s)ds$.
It is easy to check that the functional I is of C^1 with the derivative given by

$$\langle I'(u,v),(\phi,\psi)\rangle=\int_{\mathbb{R}^N}\nabla u\nabla\phi+V_1(x)u\phi dx+\int_{\mathbb{R}^N}\nabla v\nabla\psi+V_2(x)v\psi dx$$
$$-\int_{\mathbb{R}^N}f(u)\phi dx-\int_{\mathbb{R}^N}g(v)\psi dx-\int_{\mathbb{R}^N}\beta(x)v\phi dx-\int_{\mathbb{R}^N}\beta(x)u\psi dx$$

for all $\phi,\psi\in C_0^\infty(\mathbb{R}^N)$, where I' denotes the Fréchet derivative of I.
Therefore, (u,v) is a critical point of I if and only if (u,v) is a solution of system (3).

- A solution $(u,v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ of (3) is called a bound state.
- A solution is called a ground state if $(u,v) \neq (0,0)$ and its energy is minimal among the energy of all the nontrivial bound state of (3).
- A ground state satisfying $u > 0, v > 0$ is called a positive ground state.

In 2012, Chen and Zou studied the following linearly coupled Schrödinger systems

$$\begin{cases} -\Delta u + \mu u = |u|^{p-2}u + \beta v, & x \in \mathbb{R}^N, \\ -\Delta v + \nu v = |v|^{2^*-2}v + \beta u, & x \in \mathbb{R}^N, \end{cases} \tag{4}$$

where $N \geq 3, 2^* := \frac{2N}{N-2}$ and μ, ν, β are positive parameters satisfying

$$0 < \beta < \sqrt{\mu\nu}.$$

They proved that

- for $2 < p < 2^*$, there is $\bar{\mu} \in (0,1)$ such that (4) has a positive ground state solution if $0 < \mu \leq \bar{\mu}$.

Chen and Zou studied the following class of linearly coupled Schrödinger systems

$$\begin{cases} -\varepsilon^2\Delta u + V_1(x)u = |u|^{p-2}u + \beta v, & x \in \mathbb{R}^N, \\ -\varepsilon^2\Delta v + V_2(x)v = |v|^{2^*-2}v + \beta u, & x \in \mathbb{R}^N, \end{cases} \tag{5}$$

where $N \geq 3, 2 < p < 2^*$ and V_i are continuous function and satisfy

- $\inf_{x\in\mathbb{R}^N} V_i(x) \geq a_i > 0$.

Under the other assumptions and

$$0 < \beta < \sqrt{a_1 a_2},$$

they proved that (5) has a positive solution for $\varepsilon > 0$ sufficiently small.

Peng, Chen and Tang studied the following system:

$$\begin{cases} -\varepsilon^2 \Delta u + V_1(x) u = |u|^{p-2} u + \beta(x) v, & x \in \mathbb{R}^N, \\ -\varepsilon^2 \Delta v + V_2(x) v = |v|^{q-2} v + \beta(x) u, & x \in \mathbb{R}^N, \end{cases} \tag{6}$$

where $N \geq 3, 2 < p < 2^*, 2 < q \leq 2^*$, $V_i, \beta \in C(\mathbb{R}^N)$, and

- $\inf V_1(x) = 0$ and $V_2(x) \geq 0$.

Under assumption that

- $|\beta(x)|^2 \leq \theta^2 V_1(x) V_2(x)$ with $0 < \theta < 1$,

the authors proved that (6) has at least one nontrivial solution for $\varepsilon > 0$ sufficiently small.

Motivation

We can deduce the conclusions that the coupling function $\beta(x)$ or coupling constant β must be controlled by potential functions, and must at least satisfy

- $|\beta(x)|^2 \le \theta^2 V_1(x) V_2(x)$ for some $0 < \theta < 1$ or
- $0 < \beta < \sqrt{a_1 a_2}$, where $V_i(x) \ge a_i > 0$.

Motivated by the fact mentioned above, it is very natural for us to pose a question as follows:

- Can the upper control conditions of the coupling function $\beta(x)$ or coupling constant β be relaxed?

We consider the following class of linearly coupled Schrödinger system:

$$\begin{cases} -\Delta u + \lambda V_1(x)\, u = f_1(x)\, |u|^{p_1-2}\, u + \beta(x) v, & x \in \mathbb{R}^N, \\ -\Delta v + \lambda V_2(x)\, v = f_2(x)\, |v|^{p_2-2}\, v + \beta(x) u, & x \in \mathbb{R}^N, \end{cases} \qquad (E_\lambda)$$

where $N \geq 3$, $2 < p_1, p_2 < 2^*$, and $\lambda > 0$ is a parameter.

Steep potential well

For system (E_λ), we assume that V_i satisfies the following conditions:

$(V1)$ V_i is a nonnegative continuous function on $\mathbb{R}^N$;

$(V2)$ there exists $c_i > 0$ such that the set $\{V_i < c_i\} := \left\{x \in \mathbb{R}^N \mid V_i(x) < c_i\right\}$ is nonempty and has finite measure;

$(V3)$ $\Omega_i = \mathrm{int}\left\{x \in \mathbb{R}^N \mid V_i(x) = 0\right\}$ is nonempty bounded domain and has a smooth boundary with $\overline{\Omega}_i = \left\{x \in \mathbb{R}^N \mid V_i(x) = 0\right\}$;

- The potential λV satisfying conditions $(V1) - (V3)$ is usually called the steep potential well whose depth is controlled by the parameter λ.
- An interesting phenomenon is that one can expect to find the solutions which are concentrated at the bottom of the wells as the depth goes to infinity.

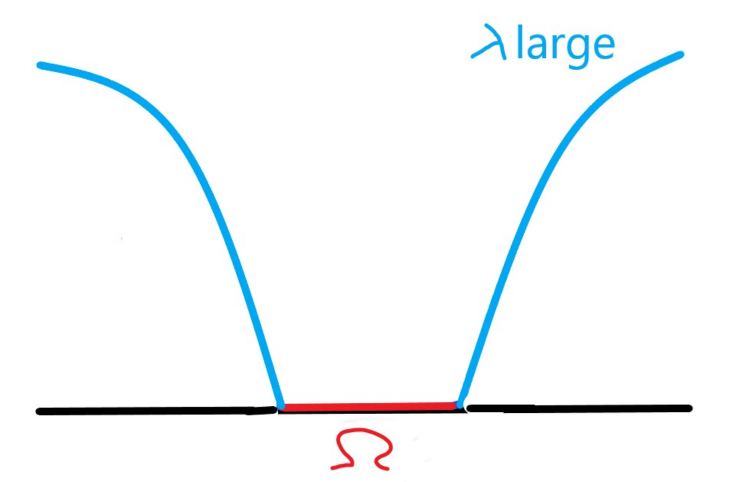

Figure: Steep potential well

Assume that the weight functions f_i and the coupling function β satisfy the following conditions:

(*F*) $0 \neq f_i \in C(\mathbb{R}^N)$ are nonnegative and $f_i(x) \leq V_i^{\frac{2^*-p_i}{2^*-2}}(x)$ for all $x \in \Omega_i^c$ for $i = 1, 2$;

(*B*) β is a nonnegative continuous functions on $\mathbb{R}^N$ and there exist $R > 0$ and

$$0 < \theta < S^2 \min\left\{ |\{V_1 < c_1\}|^{\frac{-2}{N}}, |\{V_2 < c_2\}|^{\frac{-2}{N}} \right\}$$

such that $\beta(x) < \theta$ for all $|x| \leq R$ and $\beta(x) \leq d_0\sqrt{V_1(x)\, V_2(x)}$ for all $|x| > R$, where ${}_{>}0$ and S is the best constant for the embedding of $D^{1,2}(\mathbb{R}^N)$ in $L^{2^*}(\mathbb{R}^N)$.

- One can see that the upper control condition of the coupling function $\beta(x)$ does not depend on the value $V_1(x)V_2(x)$ in the ball $\{x \in \mathbb{R}^N \mid |x| \leq R\}$.

Variational setting to our problem

- $X_i = \left\{ u \in D^{1,2}(\mathbb{R}^N) \ : \ \int_{\mathbb{R}^N} V_i(x) u^2 dx < \infty \right\}$.
- For any $\lambda > 0$, we define the Hilbert space $X_\lambda = X_1 \times X_2$ endowed with the following norm

$$\|(u,v)\|_\lambda^2 := \int_{\mathbb{R}^N} \left(|\nabla u|^2 + \lambda V_1(x) u^2 + |\nabla v|^2 + \lambda V_2(x) v^2 \right) dx.$$

- The embedding $X_\lambda \hookrightarrow H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ is continuous.
- We use the variational methods to find positive solutions of system (E_λ). Associated with system (E_λ), we define the energy functional $J_\lambda : X_\lambda \to \mathbb{R}$

$$J_\lambda(u,v) = \frac{1}{2}\|(u,v)\|_\lambda^2 - \int_{\mathbb{R}^N} \beta(x) uv dx - \frac{1}{p_1} \int_{\mathbb{R}^N} f_1(x) |u|^{p_1} dx - \frac{1}{p_2} \int_{\mathbb{R}^N} f_2(x) |v|^{p_2} dx.$$

Because the energy functional J_λ is not bounded below on X_λ, it is useful to consider the functional on the Nehari manifold

$$\mathbf{N}_\lambda = \left\{(u,v) \in X_\lambda \setminus \{(0,0)\} \mid \left\langle J'_\lambda(u,v), (u,v)\right\rangle = 0\right\},$$

where

$$\left\langle J'_\lambda(u,v), (u,v)\right\rangle = \|(u,v)\|_\lambda^2 - 2\int_{\mathbb{R}^N} \beta(x) uv dx - \int_{\mathbb{R}^N} f_1(x)|u|^{p_1} dx - \int_{\mathbb{R}^N} f_2(x)|v|^{p_2} dx.$$

- Under conditions $(V1)-(V3),(F)$ and (B), we can prove that any minimizer of J_λ constrained on $\mathbf{N}_\lambda$ is a critical point of J_λ on X_λ.

Let

$$\alpha_\lambda := \inf_{(u,v)\in \mathbf{N}_\lambda} J_\lambda(u,v)$$

Then $(u,v) \in \mathbf{N}_\lambda$ with $J_\lambda(u,v) = \alpha_\lambda$ will be a ground state of system (E_λ).

Fibering maps

It is useful to understand $\mathbf{N}_\lambda$ by the stationary points of mappings of the form

$$
\begin{aligned}
&h_{(u,v)}(t) := J_\lambda(tu, tv)(t > 0)\\
=&\frac{t^2}{2}\left\|(u,v)\right\|_\lambda^2 - t^2\int_{\mathbb{R}^N}\beta(x)uvdx - \frac{t^{p_1}}{p_1}\int_{\mathbb{R}^N} f_1(x)\left|u\right|^{p_1}dx - \frac{t^{p_2}}{p_2}\int_{\mathbb{R}^N} f_2(x)\left|v\right|^{p_2}dx.
\end{aligned}
$$

Such a map is known as the fibering map. It was introduced by Pohozaev. Clearly,

$$
\begin{aligned}
&h'_{(u,v)}(t)\\
=&t\left(\left\|(u,v)\right\|_\lambda^2 - 2\int_{\mathbb{R}^N}\beta(x)uvdx\right) - t^{p_1-1}\int_{\mathbb{R}^N} f_1(x)\left|u\right|^{p_1}dx - t^{p_2-1}\int_{\mathbb{R}^N} f_2(x)\left|v\right|^{p_2}dx.
\end{aligned}
$$

Thus, one can see that $h'_{(u,v)}(t) = 0$ if and only if $(tu, tv) \in \mathbf{N}_\lambda$; that is, points in $\mathbf{N}_\lambda$ correspond to stationary points of the maps $h_{(u,v)}$.

$$\begin{aligned}
\int_{\mathbb{R}^N} \beta(x)uvdx &\leq d_0 \int_{|x|>R} \sqrt{V_1(x)\, V_2(x)}\, |u|\, |v|\, dx + \int_{|x|\leq R} \beta(x)\, |u|\, |v|\, dx \\
&\leq \frac{d_0}{2\lambda}\left(\int_{\mathbb{R}^N} \lambda V_1(x)\, u^2 dx\right) + \frac{d_0}{2\lambda}\left(\int_{\mathbb{R}^N} \lambda V_2(x)\, v^2 dx\right) + \frac{\theta}{2}\int_{|x|\leq R} u^2 + v^2 dx \\
&\leq \frac{1}{2\lambda}\left(d_0 + \frac{\theta}{c_1}\right)\int_{\mathbb{R}^N} \lambda V_1 u^2 dx + \frac{\theta\, |\{V_1 < c_1\}|^{\frac{2}{N}}}{2S^2}\int_{\mathbb{R}^N} |\nabla u|^2 dx \\
&\quad + \frac{1}{2\lambda}\left(d_0 + \frac{\theta}{c_2}\right)\int_{\mathbb{R}^N} \lambda V_2 v^2 dx + \frac{\theta\, |\{V_2 < c_2\}|^{\frac{2}{N}}}{2S^2}\int_{\mathbb{R}^N} |\nabla v|^2 dx \\
&\leq \frac{\theta}{2S^2}\max\left\{|\{V_1 < c_1\}|^{\frac{2}{N}}, |\{V_2 < c_2\}|^{\frac{2}{N}}\right\}\|(u,v)\|_\lambda^2
\end{aligned}$$

for all $\lambda > 0$ sufficiently large. Thus,

$$\|(u,v)\|_\lambda^2 - 2\int_{\mathbb{R}^N} \beta(x)uvdx \geq \widehat{B}_0\, \|(u,v)\|_\lambda^2 \quad \text{for all } \lambda > 0 \text{ sufficiently large,} \qquad (7)$$

where

$$\widehat{B}_0 := 1 - \frac{\theta}{S^2}\max\left\{|\{V_1 < c_1\}|^{\frac{2}{N}}, |\{V_2 < c_2\}|^{\frac{2}{N}}\right\} > 0.$$

Thus, we also have

$$\alpha_\lambda = \inf_{(u,v)\in \mathbf{N}_\lambda} J_\lambda(u,v) = \inf_{(u,v)\in X\setminus\{(0,0)\}} \max_{t>0} J_\lambda(tu,tv).$$

Clearly,

$$\max_{t>0} J_\lambda(tu,tv) \geq \max_{t>0} J_\lambda(t|u|, t|v|).$$

Then

$$\alpha_\lambda = \inf_{(u,v)\in X_\lambda\setminus\{(0,0)\}} \max_{t>0} J_\lambda(tu,tv) = \inf_{(u,v)\in X_\lambda\setminus\{(0,0)\}} \max_{t>0} J_\lambda(t|u|, t|v|).$$

Hence we suppose that every ground state solution (u,v) of system (E_λ) is nonnegative. Note that if $(u,v) \neq (0,0)$ is a solution of system (E_λ), then $u \neq 0$ and $v \neq 0$. Then according to Maximum Principle, we can deduce that $u > 0$ and $v > 0$.

Main results

Theorem 2 (Y.-C. Lin–K.-H. Wang–T.-F. Wu, J. Math. Phys. 2021)

Suppose that conditions $(V1)-(V3)\,,(F)$ and (B) hold. Then system (E_λ) has a positive ground state solution for all $\lambda > 0$ sufficiently large.

Theorem 3 (Y.-C. Lin–K.-H. Wang–T.-F. Wu, J. Math. Phys. 2021)

Let (u_λ, v_λ) *be the positive solutions obtained in Theorem 2. Then* $(u_\lambda, v_\lambda) \to (u_\infty, v_\infty)$ *in* $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ *as* $\lambda \to \infty$, *where* (u_∞, v_∞) *is a positive solution of*

$$\begin{cases} -\Delta u = f_1(x)\,|u|^{p_1-2}\,u + \beta(x)v, & x \in \Omega_1, \\ -\Delta v = f_2(x)\,|v|^{p_2-2}\,v + \beta(x)u, & x \in \Omega_2, \\ u(x) = 0, & x \in \Omega_1^c, \\ v(x) = 0, & x \in \Omega_2^c. \end{cases} \tag{E_∞}$$

Thank you for your attention.