Supermodularity and Complementarity in Economics

Meng-Yu Liang

Institute of Economics, Academia Sinica

- Example 1: Find the conditions such that the optimal *a*(s)* is increasing in *s*.

$$\max_{a\in[s,1]} F(a,s) = \{P(a)(a-s) + \delta R(a)\}$$

First order condition:

$$F_a = P'(a)(a-s) + P(a) + \delta R'(a) = 0$$

Suppose the second order condition is satisfied, i.e., $F_{aa} < 0$
Take the total derivative of F.O.C we have

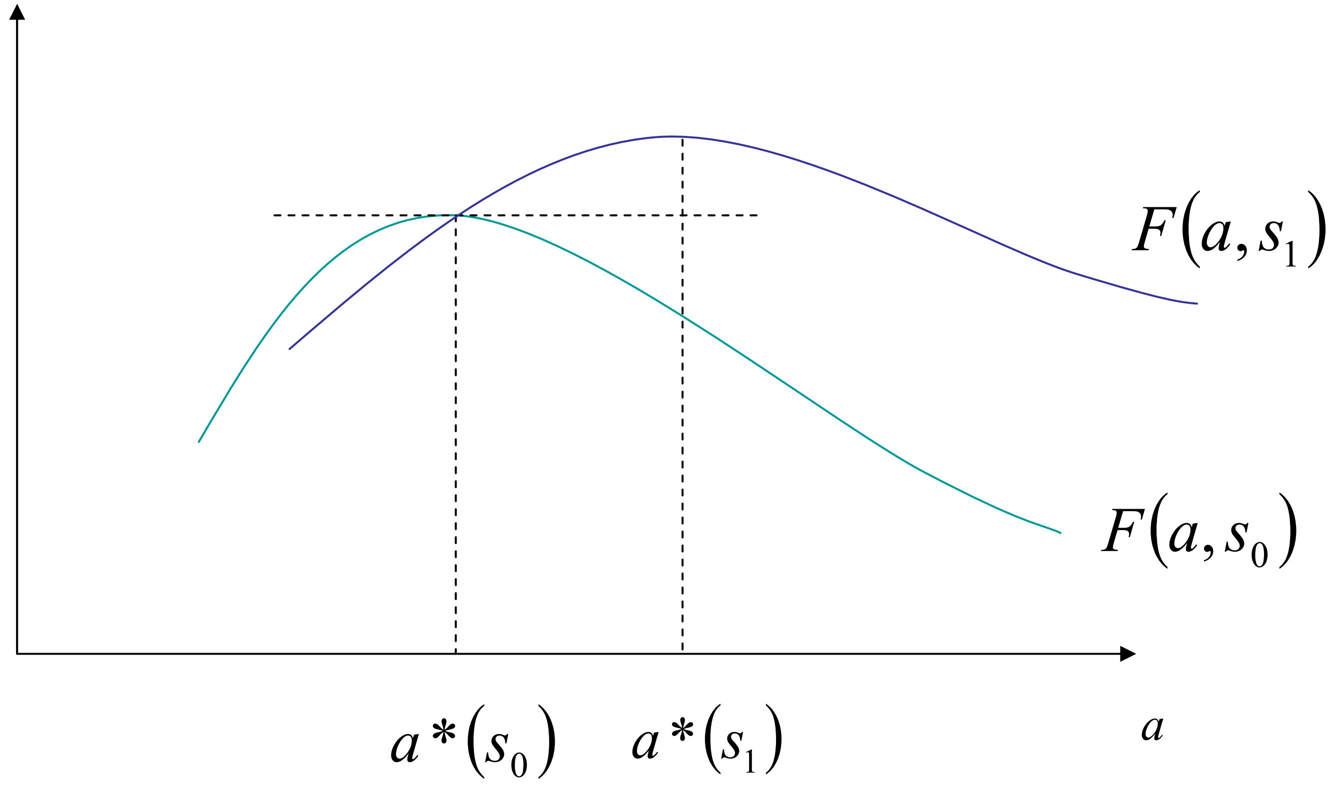
$$F_{aa}da + F_{as}ds = 0$$

$$\frac{da}{ds} = -\frac{F_{as}}{F_{aa}} = \frac{P'(a)}{P''(a)(a-s) + 2P'(a) + \delta R''(a)} > 0$$

$$F(a,s) = P(a)(a-s) + \delta R(a)$$

If F_a is increasing in s, i.e. F_{as} >0, then $a^*(s)$ is increasing in s.

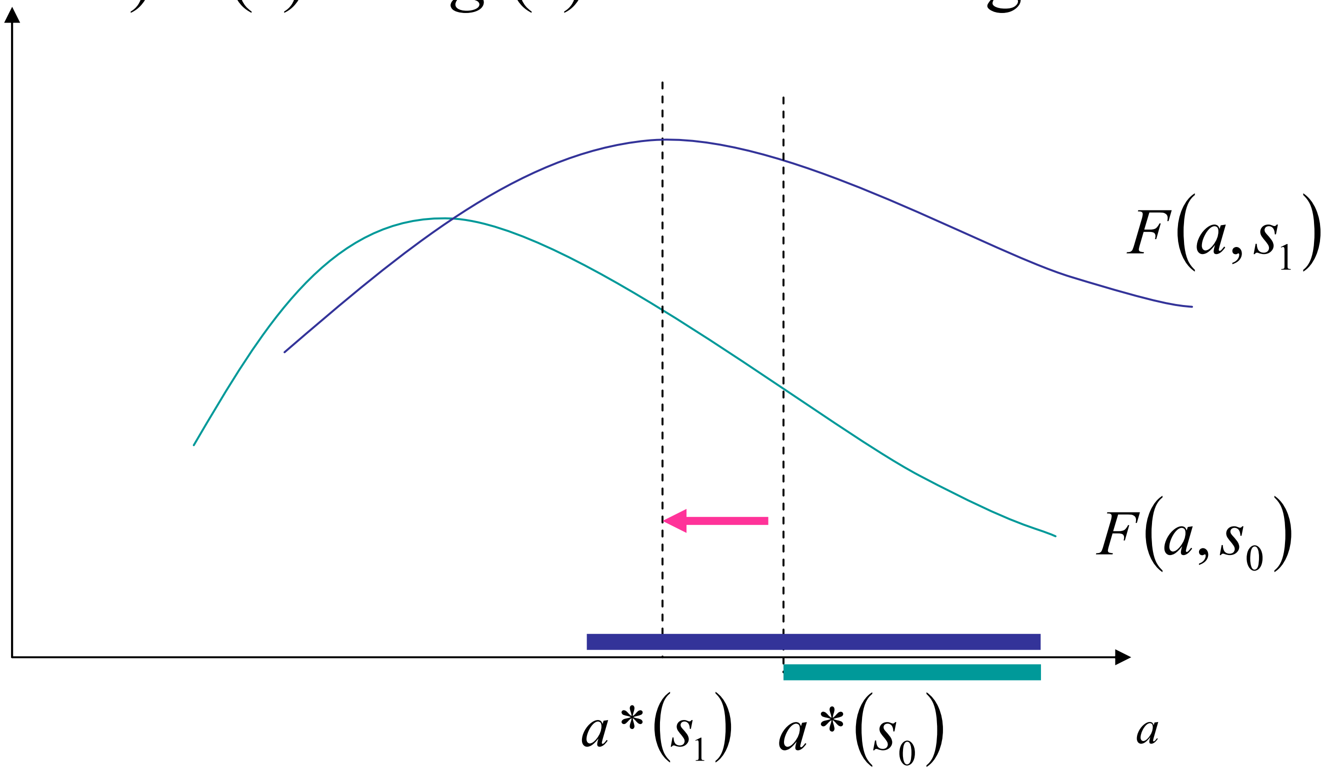
Or if a and s are complement then more s leads to more a.



$$\max_{a \in [h(s), g(s)]} F(a,s)$$

To guarantee that $a^*(s)$ is increasing in s, we need

i) $F_{as} > 0$ ii) $h(s)$ and $g(s)$ are increasing in s



Applications:

(a) Consumer theory. Is 1 a normal good in

$$\max\{U(x_1, x_2) : p_1x_1 + p_2x_2 = m\}, \text{ or}$$

$$\max\{U(x_1, (m - p_1x_1)/p_2) : x_1 \in [0, m/p_1]\}.$$

$[0, m/p_1]$ is ascending in m

The objective has incr. diffs in (x_1, m) if

$$p_2U_{21}(x_1, x_2) - p_1U_{22}(x_1, x_2) \geq 0$$

$x_1^*(p_1, p_2, m)$ is incr. in m or 1 is normal (no quasi-concavity needed!).

(d) *Growth theory with Increasing returns:*

(Amir-Mirman-Perkins 1991, Amir 1996).

2-period version of the standard Solow-Cass-Koopmans optimal growth model with increasing returns.

$$\max \sum_{t=1}^{2} u(x_t - y_t) \quad \text{subject to} \quad x_{t+1} = f(y_t) \text{ and } y_t \in [0, x_t].$$

where $u' > 0$ and $u'' < 0$, and no restrictions on f and $\delta \in (0, 1)$.

The two-period value function V_2 satisfies

$$V_2(x) = \max\{u(x - y) + \delta u[f(y)] : y \in [0, x]\}$$

Since the maximand has incr. diffs in (x, y) and the constraint set $[0, x]$ is ascending, the optimal savings correspondence $y^*(x)$ is increasing in x. (Note that no restrictions are actually needed on f.)

Infinite-horizon: By induction on the horizon length.

Similarly: $y^*(\delta, x)$ is incr. in δ.

(e) Monopoly pass-through. $\Pi(p,c) = (p-c)D(p)$ or

$$\log \Pi(p,c) = \log(p-c) + \log D(p) \ , \ p \in [c, \infty).$$

- $\log \Pi(p,c)$ has incr. diffs in (p,c) since $\partial^2 \log(p-c)/\partial p \partial c = (p-c)^{-2} \geq 0$. ($D$ need not be $\searrow$)
- But $\partial^2 \Pi(p,c)/\partial p \partial c = -D'(p) \geq 0$ iff $D' \leq 0$.
- $[c, \infty)$ is ascending
- Every selection from p^* is incr. in c.
- Let mark-up $m \triangleq p - c$ and $\widetilde{\Pi}(m,c) \triangleq \log(m) + \log D(m+c)$
- $\log \widetilde{\Pi}(m,c)$ has decr. diffs in (m,c) if D is log-concave since $\partial^2 \log D(m+c)/\partial m \partial c = [DD'' - D'^2]/D^2 \leq 0$ iff $DD'' - D'^2 \leq 0$.
- $m^*(c)$ is decr. in c, or p^* has all slopes ≤ 1 (as $p^*(c) = m^*(c) + c$).
- p^* has all slopes in $[0,1]$ and is thus continuous and single-valued. There is positive but partial pass-through.
- If D is log-convex, $\log \widetilde{\Pi}(m,c)$ has incr. diffs in (m,c), so $m^*(c)$ is incr. in c, or p^* has all its slopes ≥ 1, so pass-through $\geq 100\%$.

Lattice Theory

Let *X* be a partially order set, with the transitive, reflexive, antisymmetric order relation $\geq$.

X is a lattice if for every pair of x and y in X, we have

- $x \vee y$: the least upper bound, or **join**, of x and y,
- $x \wedge y$: the greatest lower bound, or **meet**, of x and y

exist in X.

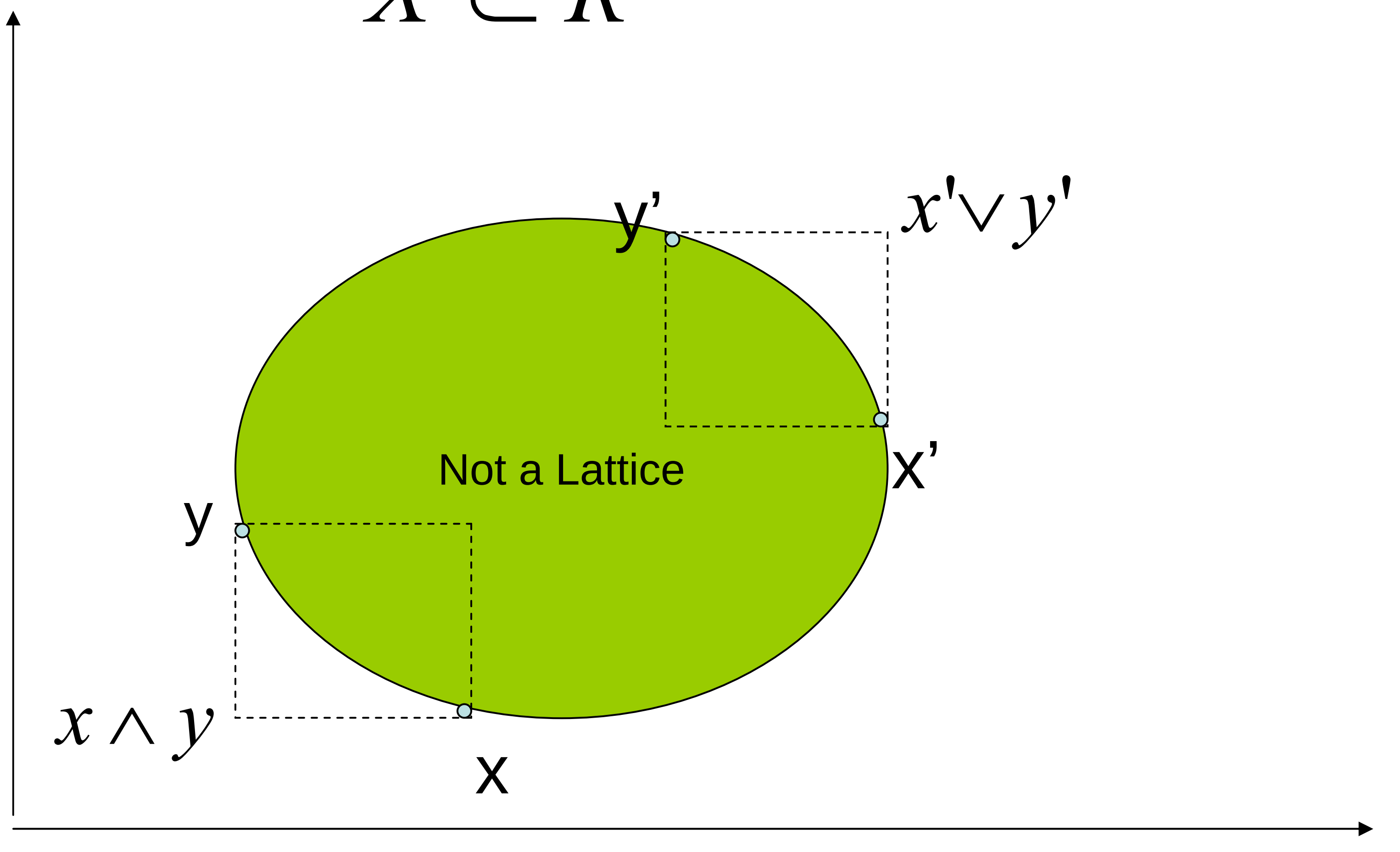
Example: the component-wise order in $X = R^2$

(1,2) $\geq$ (0,-3),

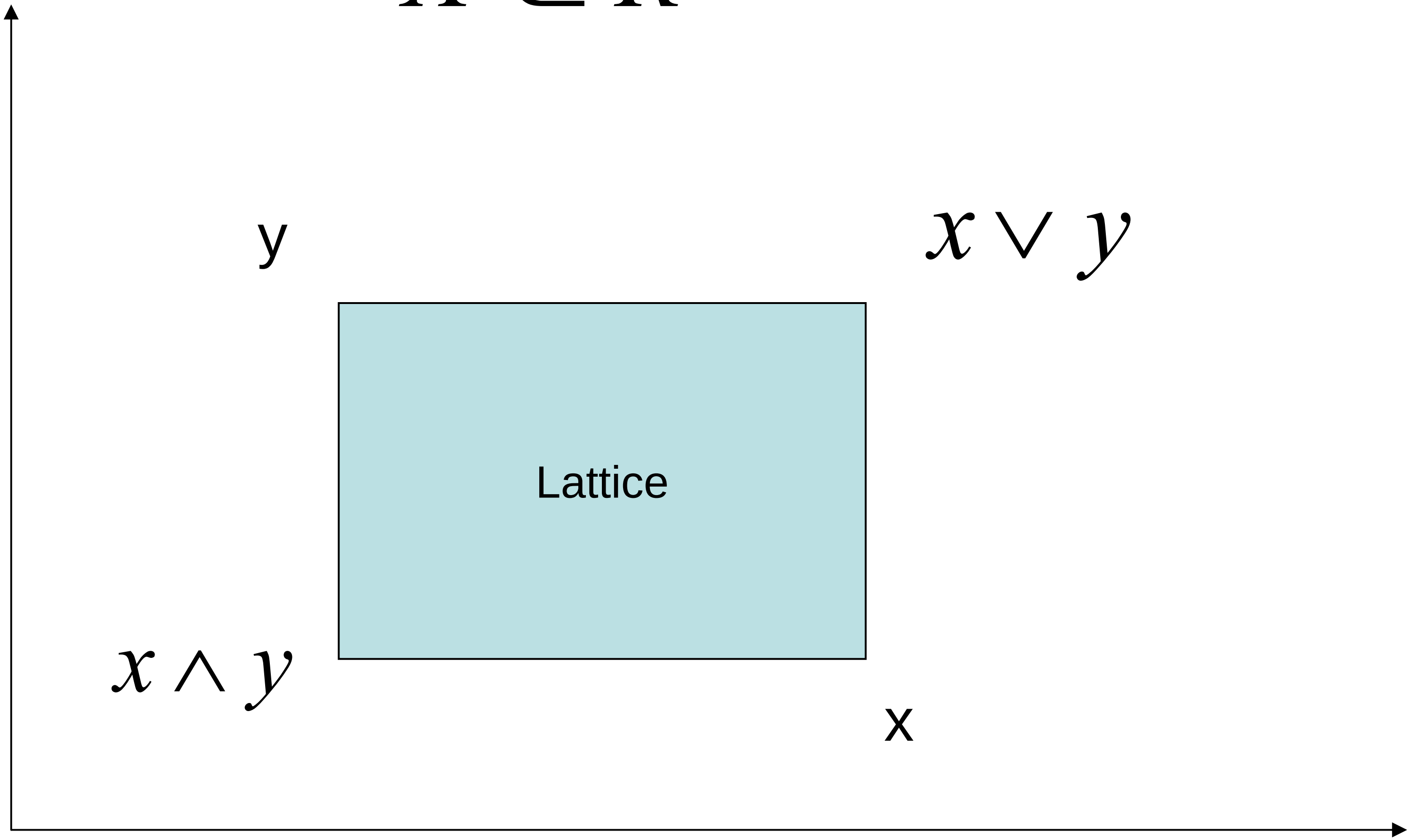
but no component-wise order for (1,2) and (3,1)

(1,2) $\vee$(3,1)=(3,2), (1,2) $\wedge$(3,1)=(1,1)

$$X \subset R^2$$



$$X \subset R^2$$



Cardinal complementarity notions

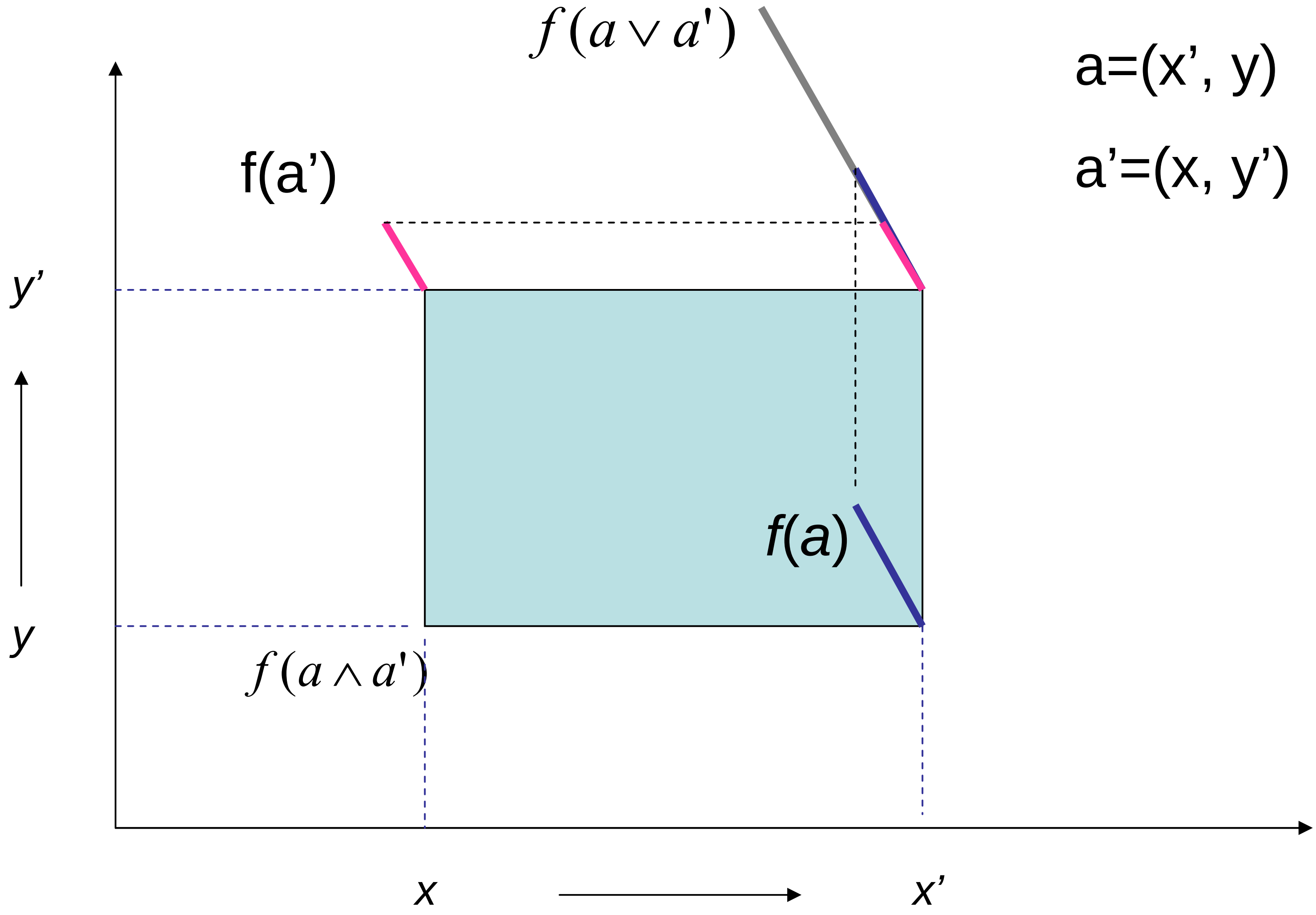
S is a poset and *A* is a lattice.

- $f : A \to R$ is **supermodular** (spm) if $\forall a, a' \in A$,
  $f(a \vee a') + f(a \wedge a') \geq f(a) + f(a')$.

- If f is C^2, f spm $\Leftrightarrow \partial^2 f(a)/\partial a_i \partial a_j \geq 0, \ \forall i \neq j$
  (all *nondiagonal* elements of the Hessian matrix of f are ≥ 0.)

- $f : A \times S \to R$ has **increasing differences** in (a, s) if $\forall a' > a, s' > s$
  $f(a', s') - f(a, s') \geq f(a', s) - f(a, s)$,
  or if the difference $f(a', \cdot) - f(a, \cdot)$ is increasing.

Spm and Increasing differences



- If f is C^2, Incr. Diffs $\Leftrightarrow \partial^2 f(a,s) / \partial a_i \partial s_j \geq 0$, for all i, j.

(no restrictions on partials $\partial^2 f / \partial a_i \partial a_j$ or $\partial^2 f / \partial s_i \partial s_j$.)

- **Special case** (common in applications) $A = S = R$:

spm in (*a*, *s*) $\Leftrightarrow$ incr. diffs in (*a*, *s*)

$$\Leftrightarrow \partial^2 f / \partial a \, \partial s \geq 0.$$

- Both properties can be checked via pairwise relations only.
- Spm and incr. diffs treat relevant variables symmetrically.

- **Interpretation of spm or incr. diffs**

  (Edgeworth) complementarity: higher values in any variables increase the marginal returns to higher values in the remaining variables.

$$a^*(s) = \arg\max \{F(a,s), a \in A_s\}$$

Theorem 1: Assume

1. F is supermodular in a for each fixed s,
2. F has increasing differences in (s,a), *and*
3. $A_s = X^{i=m}{}_{i=1}[g_i(s), h_i(s)]$ where $h_i, g_i : S \to R$ are increasing functions with $g_i \le h_i$

Then the maximal and minimal selections of $a^*(s)$ are increasing functions.

Furthermore, if 2) is strict, then every selection of $a^*(s)$ is increasing.

Example: Assortative matching (Becker 1973)

There are n women and n men to match to form n marriages.

Each sex is ranked by productivity $\{1, 2, ..., n\}$.

If i and j marry, they generate a surplus $f(i, j)$.

A matching is any list of n (straight) couples.

Question: When is $\arg\max \sum f(i, j)$ over all possible matches the assortative matching, i.e. $\{(1, 1), ..., (n, n)\}$?

Answer: If f has strictly incr. diffs.

For otherwise, there would be 2 couples (i, j) and (i', j') with (say) $i' > i$ but $j' < j$, so that by incr. diffs of f,

$$f(i', j) + f(i, j') > f(i, j) + f(i', j'),$$

a contradiction, as (i', j) and (i, j') is better than (i, j) and (i', j').

Ordinal Complementarity Condition

- Theorem 2: The conclusions of Theorem 1 holds if supermodularity is replaced by quasi-supermodularity and (strict) increasing differences by the (strict) single-crossing property

quasi-supermodularity (q-spm) and single-crossing property (SCP)

- $F : A \to R$ is q-spm, if $\forall a, a' \in A$,

  $F(a) - F(a \wedge a') \geq (>) 0 \Rightarrow$

  $F(a \vee a') - F(a') \geq (>) 0$

- $F : S \times A \to R$ has the SCP in (a; s) if

  $\forall a' > a, s' > s$,

  $F(a', s) - F(a, s) \geq (>)\ 0 \Rightarrow$

  $F(a', s') - F(a, s') \geq (>)\ 0$.

- The SCP is strict if

  $F(a', s) - F(a, s) \geq 0 \Rightarrow F(a', s') - F(a, s') > 0$.

Economic interpretation of SCP

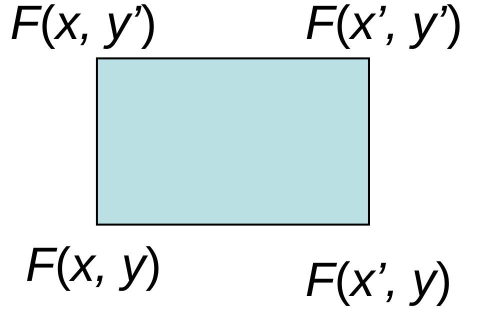
- limited complementarity: If a given increase in *a* is profitable when *s* is low, the same increase will be profitable when *s* is high.
- *F* may have SCP in (s; a) but not in (a; s) : one-way complementarity.
- *F* spm $\Rightarrow$ *F* q-spm
- *F* has incr. diffs in (s, a) $\Rightarrow$ *F* has SCP in (s; a) and in (a; s).

Properties on spm, q-spm and SCP

- F q-spm and g strictly incr $\Rightarrow g \circ F$ q-spm
- F has SCP and g strictly incr $\Rightarrow g \circ F$ has SCP.
- If $h(\cdot)$ str. incr. and $h \circ F$ is spm (incr. diffs), then F is q-spm (SCP)
- Not all q-spm functions are $= h \circ G$, with $h \nearrow$ and G spm.
- $F(\cdot)$ concave $\Leftrightarrow F(x - y)$ is spm in $(x, y) \Leftrightarrow F'' \leq 0$.
- $F(x - y)$ is spm in (x, y) and $g(\cdot) \nearrow$ and convex $\Rightarrow g \circ F$ spm in (x, y).

Log-supermodularity

- $F : A \to R$ is log-spm iff log F is spm or
  $F(a \vee a')F(a \wedge a') \geq F(a)F(a'),\ \forall a, a' \in A.$
- $F : R^2 \to R$, $F \geq 0$, is log-spm if for $(x', y') > (x, y)$
  $F(x', y')F(x, y) \geq F(x', y)F(x, y')$ or
  $$F(x', y')/F(x, y') \geq F(x', y)/F(x, y)$$
- the relative returns $F(x', \cdot)/F(x, \cdot)$ are $\nearrow$
  (or $F(\cdot, y')/F(\cdot, y)$ are $\nearrow$).
  (as opposed to absolute returns for spm).
- F spm and F log-spm are not comparable.
- F spm and F log-spm $\Rightarrow F$ q-spm.
- Log-spm survives multiplication, but not addition.



Spence-Mirrlees condition (SM)

(Milgrom and Shannon 1994)

Theorem: Let F: $R^3 \rightarrow R$ be continuously differentiable and $F_2(a,b,s) \neq 0$.

- $F(a,h(a),s)$ satisfies the SCP in (a;s) for all functions h: $R \rightarrow R$

  if and only if

  $F_1(a,b,s)/|F_2(a,b,s)|$ is increasing in s.

- $F(a,h(a),s)$ satisfies the **strict** SCP in (a;s) for all functions h: $R \rightarrow R$

  if $F_1(a,b,s)/|F_2(a,b,s)|$ is **strictly** increasing in s.

Spm games

A normal-form game (N, A_i, F_i) is spm if for each i,

(i) the action set A_i is a complete lattice.

(ii) F_i is spm in own action a_i,

(or $\partial^2 F(a^i, a^{-i})/\partial a^i_j \partial a^i_k \geq 0$ for all $j \neq k$).

(iii) F_i has incr. diffs in (a_i, a_{-i})

(or $\partial^2 F(a^i, a^{-i})/\partial a^i_j \partial a^{-i}_k \geq 0$ for all j, k.)

(No restrictions on partials of the form $\partial^2 F(a^i, a^{-i})/\partial a^{-i}_j \partial a^{-i'}_k$.)

Theorem 1 *(Tarski 1955) Let A be a complete lattice and $F : A \to A$ be increasing. Then the set E of fixed-points of F is a nonempty complete lattice. Furthermore,*

$$\overline{E} = \sup\{a : a \geq F(a)\} \text{ and } \underline{E} = \inf\{a : a \leq F(a)\}$$

Theorem 2 *(Zhou 1994) Let A be a complete lattice and $F : A \to 2^A$ be ascending. Then E is a nonempty complete lattice.*

Theorem 3 *(Topkis 1979) For a spm game, the best-reply map is ascending and the set of PSNE is a nonempty complete lattice.*

Comparative Statics of Equilibrium Points

Theorem: Assume that

1. For each $s \in S \subset R$, the game is smp, and
2. F_i has increasing differences in (a_i,s) for each a_{-i}.

Then the extremal equilibia of the game are increasing functions of s.

(c) Search. Diamond's (1980): agent i expands effort $a^i \in [0,1]$ seaching for trading partners, and has a payoff (with $s > 0$ parameter)

$$F_i(a^i, a^{-i}) = sa^i \sum_{j \neq i} a^j - C_i(a^i).$$

Since $\partial^2 F_i(a^i, a^{-i})/\partial a^i \partial a^j = s > 0, \forall\, i \neq j$, the game is spm, for any cost function.

- Since $\partial^2 F_i / \partial a^i \partial s \geq 0$, the extremal equilibria are increasing in s (which is a measure of the ease of search.)

(d) Bertrand oligopoly. $F_i(p^i, p^{-i}) = (p^i - c_i) D_i(p^i, p^{-i})$ or

$$\log F_i(p^i, p^{-i}) = \log(p^i - c_i) + \log D_i(p^i, p^{-i}), p^i \in [c_i, \infty)$$

The game is log-spm if $\log D_i$ is spm in (p^i, p^{-i}) or (say) $\forall\, j \neq 1$,

$$D^1 D^1_{p_1 p_j} - D^1_{p_1} D^1_{p_j} \geq 0.$$

Interpretation: firm i's price elasticity of demand is incr. in rivals' prices (very natural, satisfied by most demand functions).

- Since $\log(p^i - c_i)$ has incr. diffs in (p^i, c), $c = (c_1, ..., c_n)$, for each i, extremal equ. prices are $\nearrow$ in c.

(e) Cournot duopoly.

$$F_i(q_1, q_2) = q_1 P(q_1 + q_2) - C_i(q_i).$$

Since with $P' \leq 0$, we have

$\partial^2 F_i(q_1, q_2)/\partial q_1 \partial q_2 = P'(q_1+q_2)+q_1 P''(q_1+q_2) \leq 0$ for all $q_1, q_2 \geq 0$

if and only if

$$P'(z) + zP''(z) \leq 0 \text{ for all } z \geq 0,$$

The game is sbm (Novshek 1985 and Amir 1996). This conclusion is easily seen to be valid even in the n-firm case, for all n.

For $n = 2$, if (say) firm 2's decision is $-q_2$ instead of q_2, then $\partial^2 F_i(q_1, q_2)/\partial q_1 \partial(-q_2) \geq 0, i = 1, 2$, so the duopoly is a spm game.

For $n \geq 3$, the Cournot game is not spm, but there is a PSNE (Selten 1970, Novshek 1985, Kukushkin 1994).

Parametric Optimization under Uncertainty
(Susan Athey 2002)

TABLE I
SUMMARY OF RESULTS

Thm #	A: Hypothesis on u(a.e.-μ)	B: Hypothesis on f(a.e.-μ)	C: Conclusion	Corresponding comparative statics conclusion
Lem 4; Thm 1	$u(\mathbf{x},\mathbf{s}) \geq 0$ is log-spm.	$f(\mathbf{s};\theta)$ is log-spm.	$\int u(\mathbf{x},\mathbf{s})f(\mathbf{s};\theta)\,d\mu(\mathbf{s})$ is log-spm in $(\mathbf{x},\theta)$.	$\arg\max_{x\in B} \int u(\mathbf{x},\mathbf{s})f(\mathbf{s};\theta)\,d\mu(s)$ $\uparrow$ in θ and B.
Lem 5; Thm 2	$u(x,s)$ satisfies SC2 in $(x;s)$.	$f(s;\theta)$ is log-spm.	$\int u(x,s)f(s;\theta)\,d\mu(s)$ satisfies SC2 in $(x;\theta)$.	$\arg\max_{x\in B} \int u(x,s)f(s;\theta)\,d\mu(s)$ $\uparrow$ in θ and B.
Lem 7; Thm 3	$u(x,s)$ satisfies SC2 and the returns to x are quasi-concave in s.	$F(s;\theta) \geq 0$ is log-spm.	$\int u(x,s)f(s;\theta)\,d\mu(s)$ satisfies SC2 in $(x;\theta)$.	$\arg\max_{x\in B} \int u(x,s)f(s;\theta)\,d\mu(s)$ $\uparrow$ in θ and B.
Lem 8, Thm 4	$u(x,y,s)$ satisfies SM.	$f(s;\theta)$ is log-spm.	$\int u(x,y,s)f(s;\theta)\,d\mu(s)$ satisfies SM.	$\arg\max_{x\in B} \int u(x,b(x),s)f(s;\theta)\,d\mu(s)$ $\uparrow$ in θ and B for all $b:\mathbb{R}\rightarrow\mathbb{R}$.

In rows 1, 2, and 4: (A) and (B) are a *minimal pair of sufficient conditions* (Definition 4) for the conclusion (C); further, (C) is equivalent to the comparative statics result in column 4. In row 3, the same relationships hold except that (A) is not necessary for (C) to hold whenever (B) does.

Notation and Definitions. Bold variables are real vectors; italicized variables are real numbers; f is nonnegative; log-spm indicates log-supermodular (Definition 3); sets are increasing in the strong set order (Definition 1); SC2 indicates single crossing of incremental returns to x (Definition 2); and SM indicates single crossing of $x-y$ indifference curves (Section V). Arrows indicate weak monotonicity.