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NTU IO (I) : *Auction Theory and Mechanism Design II*
Groves Mechanism and AGV Mechansim

- $I+1$ players.
  - Types are drawn from independent distribution P_i on $[\underline{\theta}_i, \bar{\theta}_i]$ with strictly positive and differentiable densities p_i. The distributions are common knowledge.
  - (private values) Agent i's preferences: $u_i(x, t_i, \theta_i)$
  - Preferences are quasi-linear: For any $i \in \{1, \ldots, I\}$,

$$u_i(x, t_i, \theta_i) = V_i(x, \theta_i) + t_i,$$

  and either

$$u_0(x, t, \theta) = V_0(x, \theta) - \sum_{i=1}^{I} t_i$$

  (selft-interested principal) or

$$u_0(x, t, \theta) = \sum_{i=0}^{I} V(x, \theta)$$

  (benevolent principal), where $V_0(x, \theta) = B_0(x, \theta) - C_0(x)$, $C_0(x)$ is the principal's monetary cost from decision x and $B_0(x, \theta)$ is nonmonetary benefit.

- An allocation $y(\cdot)$ is *(ex post) efficient* if $x(\theta) \in K$ for each θ and

$$x(\theta) \text{ maximizes } \sum_{i=0}^{I} V_i(x, \theta) \text{ over } K, \text{ for all } \theta$$

- Budget Balance:

$$\sum_{i=1}^{I} t_i(\theta) \leq -C_0(x(\theta)) \text{ for all } \theta$$

- Dominant Strategy vs. Bayesian Mechanisms: Choose agent i's transfer so that agent i's payoff is the same as the total surplus of all parties up to a constant.
  - Dominant-strategy mechanism: Each agent's optimal announcement is independent of the announcements of the other agents, i.e., IC is for each agent $i = 1, \ldots, I$ and for each θ_i, $\hat{\theta}_i$ and θ_{-i},

$$u_i(y(\theta_i, \theta_{-i}), \theta_i) \geq u_i\left(y\left(\hat{\theta}_i, \theta_{-i}\right), \theta_i\right).$$

  - Bayesian mechanism: IC is

$$E_{\theta_{-i}} u_i(y(\theta_i, \theta_{-i}), \theta_i) \geq E_{\theta_{-i}} u_i\left(y\left(\hat{\theta}_i, \theta_{-i}\right), \theta_i\right)$$

- Dominant strategy mechanism is not sensitive to beliefs that players have about each other and it does not require players to compute Bayesian equilibrium strategies. However, focusing on dominant-strategy mechanism restricts the set of mechansims considerably. Thus, implementation in dominant strategies is a nice property to have if feasible, but it is not clear how much utility loss a principal should be willing to tolerate in order to have dominant strategies for the agents.
- Mookherjee and Reichelstein (1989) identify a class of models in which dominant strategy implementation involves no welfare loss relative to Bayesian implementation.

• Efficiency Theorems

- The Groves Mechanism (1973) Dominant Strategy implementation: Choose agent i's transfer so that agent i's payoff is the same as the total surplus of all parties up to a constant.
  * Let $x^*(\theta)$ maximizing $\sum_{i=0}^{I} V_i(x,\theta_i)$ denote an efficient solution for the type profile θ.
  * Define $t_i\left(\hat{\theta}\right) \equiv \sum_{\substack{j\in\{0,\ldots,I\}\\ j\neq i}} V_j\left(x^*\left(\hat{\theta}_i,\hat{\theta}_{-i}\right),\hat{\theta}_j\right) + \tau_i\left(\hat{\theta}_{-i}\right)$, where $\tau_i\left(\hat{\theta}_{-i}\right)$ is an arbitrary function of $\hat{\theta}_{-i}$.
  * Show that it is optimal for agent i to announce his true type $\left(\hat{\theta}_i=\theta\right)$ regardless of the other agents' announcements. This implies that $\left(x^*\left(\hat{\theta}\right),t\left(\hat{\theta}\right)\right)$ is a dominant strategy mechanism which yields efficient allocation. The proof is simple: Suppose that agent i strictly prefers announcing $\hat{\theta}_i$ to announcing θ_i for some types $\hat{\theta}_{-i}$ of the other agents. Then

$$\begin{aligned} & V_i\left(x^*\left(\hat{\theta}_i,\hat{\theta}_{-i}\right),\theta_i\right)+\sum_{j\neq i} V_j\left(x^*\left(\hat{\theta}_i,\hat{\theta}_{-i}\right),\hat{\theta}_j\right) \\ & > V_i\left(x^*\left(\theta_i,\hat{\theta}_{-i}\right),\theta_i\right)+\sum_{j\neq i} V_j\left(x^*\left(\theta_i,\hat{\theta}_{-i}\right),\hat{\theta}_j\right). \end{aligned}$$

    But this contradicts the fact that $x^*\left(\theta_i,\hat{\theta}_{-i}\right)$ is efficient for type profile $\left(\theta_i,\hat{\theta}_{-i}\right)$.
  * Budget balance may not be satisfied.
  * Example: Should a bridge be built? $u_i=\theta_i x+t_i$, where x is equal to 0 or 1 and θ_i is agent i's valuation or willingness to pay for the public good. With $c>0$ denoting the cost of supplying the public good, the efficient rule is

$$x^*(\theta)=\begin{cases} 1 & \text{if } \sum_{i=1}^{I}\theta_i \geq c \\ 0 & \text{otherwise} \end{cases}$$

    One Groves mechanism for this example takes the following form:

$$t_i\left(\hat{\theta}\right)=\begin{cases} \sum_{j\neq i}^{I}\hat{\theta}_j - c & \text{if } \sum_{j=1}^{I}\hat{\theta}_j \geq c \\ 0 & \text{otherwise} \end{cases}$$

    and

$$x^*\left(\hat{\theta}\right)=\begin{cases} 1 & \text{if } \sum_{i=1}^{I}\hat{\theta}_i \geq c \\ 0 & \text{otherwise} \end{cases}$$

– The AGV Mechanism: Instead of being paid the surpluses of the other agents on the basis of their reports, each agent is paid the expected value of the other agents' surpluses conditional on his own report.

$$t_i\left(\hat{\theta}\right) \equiv E_{\theta_{-i}}\left(\sum_{\substack{j\in\{0,\ldots,I\}\\ j\neq i}} V_j\left(x^*\left(\hat{\theta}_i,\theta_{-i}\right),\theta_j\right)\right)+\tau_i\left(\hat{\theta}_{-i}\right)$$

The function $\tau_i(\cdot)$ will be determined later to ensure BB.

* (x^*, t) is (Bayesian) incentive compatiable: $\hat{\theta}_i = \theta_i$ must maximize

$$E_{\theta_{-i}}\left(V_i\left(x^*\left(\hat{\theta}_i,\theta_{-i}\right),\theta_i\right)+\sum_{j\neq i} V_j\left(x^*\left(\hat{\theta}_i,\theta_{-i}\right),\theta_j\right)\right).$$

But $\hat{\theta}_i = \theta_i$ maximizes the term inside the expectation operator for all θ_{-i} and, therefore, maximizes the expectation.

* Suppose cost $C_0(x) = 0$. BB requires $\sum_{i=1}^I t_i\left(\hat{\theta}\right) = 0$. Let

$$\mathcal{E}_i\left(\hat{\theta}_i\right) \equiv E_{\theta_{-i}}\left(\sum_{j\neq i} V_j\left(x^*\left(\hat{\theta}_i,\theta_{-i}\right),\theta_j\right)\right)$$

denote the "expected externality" for agent i when he announces $\hat{\theta}_i$. Since $\mathcal{E}_i\left(\hat{\theta}_i\right)$ is the first part of the transfer to agent i and $\tau_i(\cdot)$ is supposed not to depend on $\hat{\theta}_i$, $\mathcal{E}_i\left(\hat{\theta}_i\right)$ must be paid be other agents, i.e.,

$$\begin{aligned}\tau_i\left(\hat{\theta}_{-i}\right) &= -\sum_{j\neq i}\frac{\mathcal{E}_j\left(\hat{\theta}_j\right)}{I-1}\\ &= -\frac{1}{I-1}\sum_{j\neq i} E_{\theta_{-j}}\left(\sum_{k\neq j} V_k\left(x^*\left(\hat{\theta}_j,\theta_{-j}\right),\theta_k\right)\right).\end{aligned}$$

* For $C_0(x) > 0$. BB requires $\sum_{i=1}^I t_i\left(\hat{\theta}\right) \leq -C_0\left(x\left(\hat{\theta}\right)\right)$. Consider the "fictional problem" where the agents' utility functions are

$$\tilde{V}_i(x,\theta_i) \equiv V_i(x,\theta_i) - \frac{C_0(x)}{I}$$

and the principal's cost is $\tilde{C}_0(x) \equiv 0$. We then compute the transfers $\tilde{t}_i(\cdot)$ for this fictional problem, and set $t_i(\cdot) = \tilde{t}(\cdot) - C_0\left(x^*\left(\hat{\theta}\right)\right)/I$.

* Ex post IR may not be satisfied. Need agents to sign a contract before they learn their types privately.

Exercise: A seller has a single indivisible unit of a good that he wants to sell to one of N buyers. The seller values the good at zero. The buyer i's utility is described as: $u_i(x_i, t_i, \theta_i) = \theta_i x_i + t_i$, where θ_i is a random variable uniformly distributed over $[0, 1]$, x_i denotes the probability that buyer i obtains the good, and t_i denotes buyer i's transfer. The seller's utility is $u_0 = 0 \cdot x_0 + t_0 = t_0$. Any feasible vector x must satisfy: $x_i \geq 0$ for all $i = 0, \cdots, N$, and $\sum_{i=0}^N x_i = 1$.

1. Characterize a Pareto-efficient allocation $x^*(\theta)$. (Hint: You may find it useful to define $z_{-i}(\theta_{-i}) \equiv \max_{j \neq i} \{\theta_j\}$ and work with it.)

2. Design a Groves mechanism that implements $x^*(\theta)$.

3. Now, impose the IR constraints, $u_i \geq 0$, and "BB" constraint, $\sum_{i=0}^{N} t_i \leq 0$. Find the Groves mechanism implementing $x^*(\theta)$ that yields the highest $t_0(\theta)$.

4. Find the Bayesian Nash equilibrium for the second-price auction game. (i.e. the highest bidder gets the object and pays the second highest bidding price.)

5. Give an economically meaningful explanation of the relationship between (3) and (4).

1) Find the first best decision $x^*(\theta)$

$$\begin{aligned} x^*(\theta) &\in \arg\max \Sigma_{i=0}^{N} v_i(x_i, \theta_i) \text{ s.t. } \Sigma_{i=0}^{N} x_i = 1 \\ &= \arg\max \Sigma_{i=0}^{N} \theta_i x_i \text{ s.t. } \Sigma_{i=0}^{N} x_i = 1 \end{aligned}$$

$\Sigma_{i=0}^{N} \theta_i x_i$ is maximized when all the x goes to the individual with largest value of θ.
Hence,

$$x_i^*(\theta) = \begin{cases} 1 & \text{if } \theta_i \geq z_i(\hat{\theta}_{-i}) \\ 0 & \text{o.w.} \end{cases} \quad \text{for } i = 0, 1, \ldots, n$$

2). Groves Mechanism: By definition $t_i = \Sigma_{j \neq i} v_j \left(x^*\left(\hat{\theta}_i, \hat{\theta}_{-i}\right), \hat{\theta}_j\right) + \tau_i\left(\hat{\theta}_{-i}\right)$ and
$x_i^*\left(\hat{\theta}\right) = \begin{cases} 1 & \text{if } \theta_i \geq z_i(\hat{\theta}_{-i}) \\ 0 & \text{o.w.} \end{cases}$ $\forall i$, we have payoff to i when reporting $\hat{\theta}_i$

$$\begin{aligned} x_i^*\left(\hat{\theta}\right)\theta_i + t_i\left(\hat{\theta}\right) &= \begin{cases} \theta_i + \Sigma_{j \neq i} v_j \left(x^*\left(\hat{\theta}_i, \hat{\theta}_{-i}\right), \hat{\theta}_j\right) + \tau_i\left(\hat{\theta}_{-i}\right) & \text{if } \hat{\theta}_i \geq z_i(\hat{\theta}_{-i}) \\ \Sigma_{j \neq i} v_j \left(x^*\left(\hat{\theta}_i, \hat{\theta}_{-i}\right), \hat{\theta}_j\right) + \tau_i\left(\hat{\theta}_{-i}\right) & \text{o.w.} \end{cases} \\ &= \begin{cases} \theta_i + \tau_i\left(\hat{\theta}_{-i}\right) & \text{if } \hat{\theta}_i \geq z_i(\hat{\theta}_{-i}) \\ z_i(\hat{\theta}_{-i}) + \tau_i\left(\hat{\theta}_{-i}\right) & \text{o.w.} \end{cases} \end{aligned}$$

An agent would only prefer $\hat{\theta}_i \geq z_i(\hat{\theta}_{-i})$ if $\theta_i \geq z_i(\hat{\theta}_{-i})$.
An agent would only prefer $\hat{\theta}_i < z_i(\hat{\theta}_{-i})$ if $\theta_i < z_i(\hat{\theta}_{-i})$.
Hence, it is the best interest of the agent to set $\theta_i = \hat{\theta}_i$. All agents tell the truth.

3) Seller : $\max_{t(\theta)} t_0(\theta)$ s.t. $\Sigma_{i=0}^{N} t_i(\theta) = 0$ and $u_i(x_i, t_i, \theta_i) \geq 0$.
Without loss of generality, suppose $\theta_1 < \theta_2 < \ldots < \theta_N$.
Seller's maximization problem becomes:

$$\max_{\tau_i(\theta_{-i})} -(\Sigma_{i=1}^{N-1} z_i(\theta_{-i}) + \tau_i(\theta_{-i})) - \tau_N(\theta_{-N})$$

$$\begin{gathered} \text{s.t.} z_i(\theta_{-i}) + \tau_i(\theta_{-i}) \geq 0 \text{ for } i = 1, \ldots, N-1 \\ \theta_N + \tau_N(\theta_{-N}) \geq 0 \end{gathered}$$

In order to maximize this expression, make the IR constrains for the low valuation types bind. This implies $\tau_i(\theta_{-i}) = -z_i(\theta_{-i})\ \forall i = 1, ..., N-1$.

Hence, the question becomes

$$\max_{\tau_i(\theta_{-i})} -\tau_N(\theta_{-N})$$

$$\text{s.t. } \theta_N + \tau_N(\theta_{-N}) \geq 0$$

Since $\tau_N(\theta_{-N})$ can not depend on θ_N, the minimum $\tau_N(\theta_{-N})$ to satisfy IR for all possible θ is $-z_N(\theta_{-N})$.

Hence, $t_0(\theta) = -\tau_N(\theta_{-N}) = z_N(\theta_{-N}) = \theta_{N-1}$ and $\tau_i(\theta_{-i}) = -z_i(\theta_{-i})\ \forall i = 1, ..., N$.

5) Under the mechanism in (3), all agents bid their valuation. The agent with the highest valuation receives the good and pays a transfer payment equals to the 2nd highest bid. All other agents pay nothing. This is exactly a second price auction. This shows that the second price auction is the most profitable Groves mechanism for the seller which satisfies the IR and BB constraints. With uncertainty, the seller has to pay information rent $\theta_N - \theta_{N-1}$ for the buyer with the highest valuation for telling the truth. Hence, the seller can only receive the highest virtual surplus $\theta_N - (\theta_N - \theta_{N-1}) = \theta_{N-1}$.