NTU IO (I) : Classnote 04 Durable good monopoly (Coasian Dynamic game)

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The market is populated by a continuum of infinitely-lived consumers, indexed by $q \in I = [0, 1]$. All consumers are risk neutral and have the same discount rate r. Each consumer wishes to possess at most one unit of the durable good. We assume that the flow benefit of the services consumer q derives from owning one unit of the durable good is described by the following inverse demand function:

$$F(q) = \begin{cases} a, & \text{if } q \in [0, \hat{q}] \\ b, & \text{if } q \in (\hat{q}, 1] \end{cases}, \text{ where } a > b > 0 \text{ and } 0 < \hat{q} < 1.$$

Let $f(q)$ denote consumer q's willingness to pay for the privilege of a one-time opportunity of acquiring one unit of the durable good. That is,

$$f(q) = \int_0^\infty F(q) e^{-rs} \mathrm{d}s = \begin{cases} \overline{v}, & \text{for } q \in [0, \hat{q}] \\ \underline{v}, & \text{for } q \in (\hat{q}, 1], \end{cases}$$

where $\overline{v} = \frac{a}{r}$ and $\underline{v} = \frac{b}{r}$. Thus, if the price at time t is p, then by purchasing or selling a unit of the durable good (and never transacting thereafter), consumer q can derive a net surplus of $e^{-rt}(f(q) - p)$ or $e^{-rt}(p - f(q))$, respectively.

A consumer is allowed to access the market as often as she wishes. Consumers seek to maximize the present value of their expected net surplus over all possible trading decisions, as a function of their holding status.

The market is served by a monopolist whose marginal cost of production, c, is constant and less than $\frac{b}{r}$. Without loss of generality, we normalize c to zero. The monopolist seeks to maximize the net expected present value of profits, using the same discount rate as consumers, r.

Sales occur at times $t = 0,\ z,\ 2z, \ldots,\ nz, \ldots$, and neither the monopolist nor consumers are allowed to trade at any time $t \in (nz, (n+1)z)$. We will refer to the time $t = nz$ as "period n". The timing of play within each period is as follows. Before trade, the monopolist selects a price, p. Then consumers can trade (buy or sell) with the monopolist at the price p, or choose not to trade. After trade occurs, a time interval of length z elapses, after which play is repeated.Marginal cost for the monopolist is normalized to 0. Monopolist offers the durable good for sale at discrete moments in time. $n = 0, 1, 2, \ldots$

Common discount factor $\delta = e^{-rz}$; r is an interest rate and z is the time length between two successive offers.

Let Q_n be the set that consumers accept the monopolist's offer in period n. Assume Q_n is measurable. Since consumers are anonymous, a histry in period n is

$(p_0, |Q_0|, p_1, |Q_1|, \ldots, p_{n-1}, |Q_{n-1}|)$ for the seller and

$(p_0, |Q_0|, p_1, |Q_1|, \ldots, p_{n-1}, |Q_{n-1}|, p_n)$ for consumers who still in the market.

Staionary equilibrium is a subgame perfect equilibrium in which every consumer's strategy depends on current price only.

More property on the stationary equilibrium (weak Markov EQ) and (P, t, R).

1. Skimming Property: Suppose that the buyer accepts price p_t at date t when he has valuation v. Then he accepts price p_t with probability 1, when he has valuation $v' > v$.

   Proof:: $h_t = (p_0, p_1, \ldots, p_{t-1})$, and if q accepts p_t then

$$f(q) - p_t \geq \delta V_q(h_t, p_t)$$

$$V_q(h_n, p_n) = \max_{s \in \{0,1\}} s(f(q) - p_n) + (1 - s)\delta V_q(h_{n+1}, p_{n+1})$$

   If $f(q') > f(q)$ then $V_{q'} > V_q$ since q' can always adopt q's strategy after date $t + 1$. This implies

$$(V_{q'} - V_q) \leq f(q') - f(q)$$

$$(1 - \delta) V_q < (1 - \delta) V_{q'}$$

   Hence

$$f(q') - \delta V_{q'} \geq f(q) - \delta V_q > p_t$$

   Skimming Property +Stationary assumption we have

$$R(q) = \max_{q' \in (q,1]} \{P(q')(q' - q) + \delta R(q')\}$$

$$t(q) = \min T(q), \text{ and } T(q) \text{ is the argmax } \{\cdot\}$$

$$P(q) = (1 - \delta) f(q) + \delta P(t(q))$$

2. $P(q)$ has to be upper-semi continuous. (Otherwise maximum doesn't exists)

   This also requires $f(q)$ to be left continuous.

3. $T(q)$ has to be a continuous correspondence and $t(q) = \min T(q)$.

   This implies $P(q)$ is well defined.

4. For gap case, i.e., $f(1) > 0$, we have unique (P, R, T). However, from (P, R, T), we can construct more than 1 subgame perfect equilibrium.

After you compute (P, R, t), what is a stationary equilibrium associated with (P, R, t) :

Consumer q's strategy: Accept p_n if and only if $p_n \leq P(q_n)$.

The seller's strategy in period n depends on $q = \sum_{i=0}^{n-1} |Q_i|$ and previous price offered p_{-1}: If $p_{-1} \geq P(q)$ then $p_n = P(t(q))$. If $P(t(q)) < p_{-1} < P(q)$, then the monopolist should play a mixed strategy such that the expected price $\bar{p}$, satisfies:

$$f(q) - p_{-1} \geq \delta (f(q) - \bar{p}), \text{ but}$$

$$f(q') - p_{-1} \leq \delta (f(q') - \bar{p}), \text{ for all } q' \in (q, 1].$$

Lemma 1 *In every stationary subgame perfect equilibrium* $P(q) \geq f(1)$.

Proof. Let $p = \inf\{$ prices which are rejected with positive probability after any histry in any stationary subgame perfect equilibrium$\}$

Note that $p > -\infty$.

For example, if the sum of monopoly and consumer surplus is bounded by

$$\int_0^1 f(z)\,\mathrm{d}z \leq 1,$$

then we know $p \geq -1$.

Now, suppose to the contrary that $p < f(1)$. Let the monopolist charge $(1-\delta) f(1) + \delta p = p'$. Then everybody must accept p'.

$$f(q) - p' \geq \delta(f(q) - p)$$

$$p' \leq (1-\delta) f(q) + \delta p$$

which holds for any $q \in [0, 1]$. This yields a contradiction. ∎

A simple two-types example: Demand Curve:

$$f(q) = \begin{cases} \overline{v} & \text{if} \quad q \in [0, \hat{q}] \\ \underline{v} & \text{if} \quad q \in (\hat{q}, 1] \end{cases}$$

and $\hat{q}\overline{v} < (1-\hat{q})\underline{v}$ (This condition ensures that the monopolist prefers not to serving the whole market in one-shut game.)

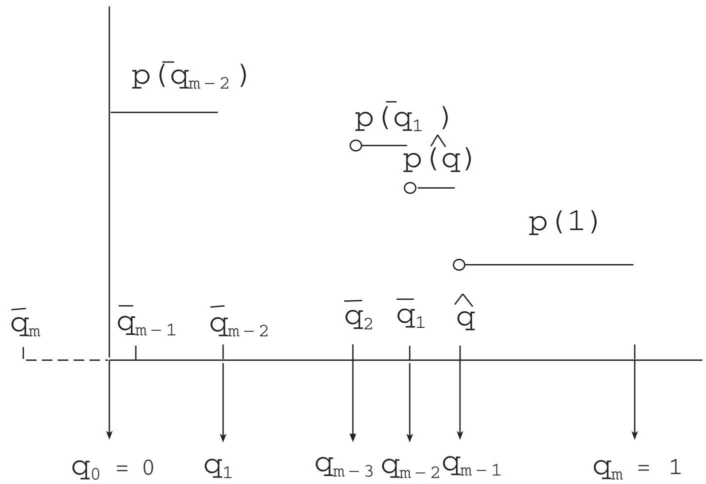
Equilibrium path $\{q_n, P(q_n)\}_{n=1}^{m}$.

Buyers' strategy: If $p \in [0, P(q_i)]$, then the consumers who are still in the market and satisfy $q \in [0, q_i]$ buy one unit of goods.

The monopolist strategy:

If $p_{k-1} = P(q_{j-1})$ then $p_k = P(q_j)$.

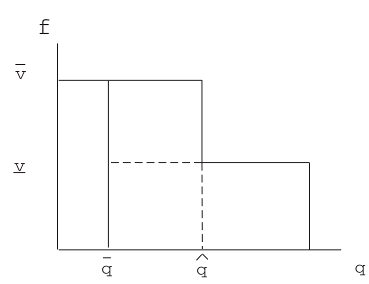
If $p_{k-1} \in (P(q_{j-1}), P(q_j))$ then $p_k = P(q_j)$ with probability π and $p_k = P(q_{j+1})$ with probability $1 - \pi$, where π satisfies $p_{k-1} = (1-\delta)\overline{v} + \delta[\pi\, P(q_j) + (1-\pi)P(q_{j+1})]$



$$\begin{cases} R(q) &= \max_{q' \in [q,1]} \{P(q')(q'-q) + \delta\, R(q')\} \\ t(q) &= \arg\max_{q' \in [q,1]} \{P(q')(q'-q) + \delta\, R(q')\} \\ P(q) &= (1-\delta) f(q) + \delta\, P(t(q)) \end{cases}$$

Let

$$\bar{q} = \inf\{q : \arg\max_{q' \in (q,1)} (q'-q) f(q') \;=\; 1\}$$



Since $\bar{q} < \hat{q}$, we know that if $q \in (\bar{q}, 1]$ then we have

$$t(q) = 1$$

$$P(1) = \underline{v}$$

$$R(q) = (1-q)\underline{v}$$

Hence, there are finite periods, m, at which the monopolist will charge $\underline{v}$ to clear the market. At period $m-1$, consumer knows that the next period price is $\underline{v}$. Hence, the consumer $q' < \hat{q}$ will buy in this period if $p_{m-1} \leq (1-\delta)\overline{v} + \delta\underline{v}$. Hence, the monopolist will set $p_{m-1} = (1-\delta)\overline{v} + \delta\underline{v}$. There exists $\bar{q}_2$ such that for $q \in (\bar{q}_2, \bar{q}]$

$$R(q) = \max\left\{ \max_{q' \in [q, \bar{q}]} \left\{ \Big((1-\delta)\overline{v} + \delta\,\underline{v}\Big)(q'-q) + \delta(1-q')\underline{v} \right\}, \underline{v}(1-q) \right\}$$

Hence, $t(q) = 1$ for $q \in (\bar{q}_1, 1]$ and $t(q) = \hat{q}$ for $q \in (\bar{q}_2, \bar{q}_1]$, and

$$P(\hat{q}) = (1-\delta)\overline{v} + \delta\underline{v}$$

and $\bar{q}_1$ satisfies

$$R(\bar{q}_1) = \underline{v}(1-\bar{q}_1) = \Big((1-\delta)\overline{v} + \delta\,\underline{v}\Big)(\hat{q} - \bar{q}_1) + \delta(1-\hat{q})\underline{v}$$

This implies that

$$\bar{q}_1 = \frac{\overline{v}}{\overline{v} - \underline{v}}\,\bar{q} - \frac{\underline{v}}{\overline{v} - \underline{v}}$$

And

$$\bar{q}_1 - \hat{q} = -(1-\hat{q})\frac{\underline{v}}{\overline{v} - \underline{v}} \tag{1}$$

From the above argument we know that at period $m-k-1, k \geq 2$, we have

$$\begin{aligned} R(\bar{q}_k) &= P(\bar{q}_{k-1})(\bar{q}_{k-1} - \bar{q}_k) + \delta\, R(\bar{q}_{k-1}) \\ &= P(\bar{q}_{k-2})(\bar{q}_{k-2} - \bar{q}_k) + \delta\, R(\bar{q}_{k-2}) \qquad (2) \\ t(q) &= \bar{q}_k \qquad \text{for} \quad q \in (\bar{q}_{k+2}, \bar{q}_{k+1}] \\ P(\bar{q}_k) &= (1-\delta)\overline{v} + \delta P(\bar{q}_{k-1}) \end{aligned}$$

Let $q_0 = 0$, $q_j = \bar{q}_{m-j-1}$ for $j = 1, \cdots, m-2$ and $q_{m-1} = \hat{q}$. Then we can defind a Weak Markov equalibrium. Hence, the remaining work is to find $\bar{q}_k$ for $k = 2, \cdots, m$, where m satisfies $\bar{q}_{m-1} \geq 0 > \bar{q}_m$

From equation 2, we have

$$
\begin{aligned}
(P(\bar{q}_{k-1}) - P(\bar{q}_{k-2}))\bar{q}_k = ((P(\bar{q}_{k-1}) - P(\bar{q}_{k-2}))\bar{q}_{k-1} \\
+ P(\bar{q}_{k-2})(\bar{q}_{k-1} - \bar{q}_{k-2}) + \delta(R(\bar{q}_{k-1} - R(\bar{q}_{k-2}))
\end{aligned}
\tag{3}
$$

Claim 1: $P(\bar{q}_k) = \overline{v} - \delta^{k-1}(\overline{v} - \underline{v})$

Proof of Claim1:

$$P(\bar{q}_k) = (1-\delta)f(\bar{q}_k) + \delta P(\bar{q}_{k-1})$$

$$
\begin{aligned}
P(\hat{q}) &= (1-\delta)\overline{v} + \delta\underline{v} = \overline{v} - \delta(\overline{v} - \underline{v}) \\
P(\bar{q}_1) &= (1-\delta)\overline{v} + \delta\left(\overline{v} - \delta(\overline{v} - \underline{v})\right) \\
&= \overline{v} - \delta^2(\overline{v} - \underline{v}) \\
&\dots \\
P(\bar{q}_k) &= (1-\delta)\overline{v} + \delta\left(\overline{v} - \delta^k(\overline{v} - \underline{v})\right) \\
&= \overline{v} - \delta^{k+1}(\overline{v} - \underline{v}) \qquad \Box
\end{aligned}
$$

From claim1, we have $P(\bar{q}_k) - P(\bar{q}_{k-1}) = \delta^{k-1}(1-\delta)(\overline{v} - \underline{v})$
and $P(\bar{q}_k) - \delta\, P(\bar{q}_{k-1}) = (1-\delta)\overline{v}$
From equation 2, we have

$$
\begin{aligned}
R(\bar{q}_{k-1}) - R(\bar{q}_{k-2}) &= P(\bar{q}_{k-3})(\bar{q}_{k-3} - \bar{q}_{k-1}) + \delta\, R(\bar{q}_{k-3}) \\
&\quad - \Big(P(\bar{q}_{k-3})(\bar{q}_{k-3} - \bar{q}_{k-2}) + \delta\, R(\bar{q}_{k-3})\Big) \\
&= -P(\bar{q}_{k-3})(\bar{q}_{k-1} - q_{k-2})
\end{aligned}
$$

Hence equation 3 becomes

$$
\begin{aligned}
\bar{q}_k &= \bar{q}_{k-1} + (\bar{q}_{k-1} - \bar{q}_{k-2}) \frac{P(\bar{q}_{k-2}) - \delta\, P(\bar{q}_{k-3})}{P(\bar{q}_{k-1}) - P(\bar{q}_{k-2})} \\
\bar{q}_k - \bar{q}_{k-1} &= (\bar{q}_{k-1} - \bar{q}_{k-2}) \frac{\overline{v}}{\delta^{k-1}(\overline{v} - \underline{v})} \\
&\cdots \\
&= (\bar{q}_1 - \hat{q})(\frac{\overline{v}}{\overline{v} - \underline{v}})^{k-1} \delta^{-(k-1+k-2+\cdots+1)} \\
&= (\bar{q}_1 - \hat{q})(\frac{\overline{v}}{\overline{v} - \underline{v}})^{k-1} \delta^{-k(k-1)/2} \\
&= -(1 - \hat{q}) \frac{\underline{v}}{\overline{v} - \underline{v}} (\frac{\overline{v}}{\overline{v} - \underline{v}})^{k-1} \delta^{-k(k-1)/2}
\end{aligned}
$$

$$
\begin{aligned}
\bar{q}_k &= (\bar{q}_k - \bar{q}_{k-1}) + (\bar{q}_{k-1} - \bar{q}_{k-2}) + \cdots + (\bar{q}_1 - \hat{q}) + \hat{q} \\
&= \hat{q} - (1 - \hat{q})(\frac{\underline{v}}{\overline{v} - \underline{v}}) \sum_{j=1}^{k} (\frac{\overline{v}}{\overline{v} - \underline{v}})^{j-1} \delta^{-j(j-1)/2}
\end{aligned}
$$

For a general gap case, i.e., $f(1) > c$, let $\bar{q} = \inf\{q : \arg\max_{q' \geq q} (q' - q)\, f(q') = (1 - q)\, f(1)\}$.

Assumption 1: Demand $f(q)$ is Lipschitz at $q = 1$, i.e. $\exists L \in [0, \infty),\ f(q) - f(1) \leq L(1 - q)$.

Lemma 2 *If $f(q)$ satisfies A1 then $\exists\ \bar{q} < 1$.*

Proof. Define $\tilde{q} = 1 - \frac{f(1)}{L}$, then for some $q \geq \tilde{q}$

$$
\begin{aligned}
(q' - q)\, f(q') &\leq (q' - q)\,(L(1 - q') + f(1)) \\
&= f(1)(1 - q) + (1 - q')(L(q' - q) - f(1))
\end{aligned}
$$

$L(q' - q) < L(1 - q) \leq L(1 - \tilde{q}) = f(1)$. Hence, $(q' - q)\, f(q') \leq f(1)(1 - q)$. This implies $\bar{q}$ $\leq \tilde{q}$ ■

With the above lemma, we can construct the equilibria from the tail $(\bar{q} < 1, 1]$.

Example 2: no-gap case Suppose $F(q) = 1 - q$

If t, P are linear and R is quadratic, then any solution to $F(q)$ in $[q, 1]$ is a re-scale of F in [0, 1]. Hence, all solution is proportion to $1 - q$. Instead of assuming $t(q) = a_0 q + b_0$,

$P(q) = a_1 q + b_1 q, R(q) = a_2 q^2 + b_2 q + c$, we assume

$$t(q) = \beta(1-q)$$
$$P(q) = \alpha(1-q)$$
$$R(q) = \frac{r}{2}(1-q)^2$$

solve for β, α and γ and check $\alpha = \sqrt{1-\delta}$. Hence, $\lim_{\delta\to 1} P(0) = 0$. Therefore, the Coase Conjecture holds in this equilibrium.

Note 1: There are other solutions. For example, t,P are not continuous functions. Note 2: For the no-gap case, there is no end period as the initial step to apply backward induction construction.

0.1 Relation with the one-sided incomplete information bargaining

Durable Goods Monopoly	$\leftrightarrow$	Bargaining with one sided incomplete information
subgame perfect equilibrium	$\leftrightarrow$	sequential equilibrium
Demand Curve $f(q)$	$\leftrightarrow$	Distribution of the buyer's valuation
quantity sold	$\leftrightarrow$	Probability of sale

perfect information $\longrightarrow$no inefficiency

incomplete information $\longrightarrow$inefficiency

need time burning the surplus to reveal buyer's (or seller's) type.

Coase Conjecture: If bargaining time goes 0, no inefficiency.

0.2 Useful Readings:

Ausubel, L. and R. Deneckere (1989), "Reputation in Bargaining and Durable Goods Monopoly," *Econometrica*, **57**, 511–531.

Deneckere, Ray and Meng-Yu Liang (2006) "Bargaining with Interdependent Values,"Econometrica, 74(5), 1309-1364.

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Gul, F., H. Sonnenschein, and R. Wilson (1986), "Foundations of Dynamic Monopoly and the Coase Conjecture," *Journal of Economic Theory*, **39**, 155–190.