NOTES OF 11 AVRIL 2018 : INTRODUCTION TO PERVERSE SHEAVES

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1. Aim

The aim is to explain the definition of $\mathrm{Perv}(X) \subseteq \mathrm{D}_c^b\left(X, \overline{\mathbf{Q}}_\ell\right)$. Here $\mathrm{Perv}(X)$ is an abelian subcategory of the triangulated category $\mathrm{D}_c^b\left(X, \overline{\mathbf{Q}}_\ell\right)$. For a quasi-projective scheme over a finite or an algebraically closed field.

Why perverse sheaves ?

1. The intersection cohomology provides for singular schemes a cohomology theory which satisfies the Poincaré duality.
2. The Riemann–Hilbert correspondence expresses in terms of perverse sheaves.

2. t-structure

Définition 2.1. — Let D be a triangulated category. A pair $\left(D^{\geq 0}, D^{\leq 0}\right)$ is called a **t-structure** of D if $D^{\geq 0}, D^{\leq 0}$ are full subcategories of D (we write then $D^{\leq n} = D^{\leq 0}[-n]$) and the following conditions are satisfied

1. $\mathrm{Hom}\left(D^{\leq 0}, D^{\geq 1}\right) = 0$
2. $D^{\leq 0} \subseteq D^{\leq 1}$, $D^{\geq 1} \subseteq D^{\geq 0}$
3. For all $E \in D$, there exists a distinguished triangle in D

$$A \xrightarrow{u} E \xrightarrow{v}\to A[1]$$

such that $A \in D^{\leq 0}$ and $B \in D^{\geq 1}$.

Exemple 2.2. — If $\mathscr{A}$ is an abelian category with derived category $D = \mathrm{D}(\mathscr{A})$, then

$$D^{\leq 0} = \left\{X \in D \;;\; \mathrm{H}^i(X) = 0,\ \forall i > 0\right\} \subseteq D$$
$$D^{\geq 0} = \left\{X \in D \;;\; \mathrm{H}^i(X) = 0,\ \forall i < 0\right\} \subseteq D$$

is a t-structure, which is called **standard t-structure**. For the condition 3, given any object $E \in D$, we let

$$A = \tau_{\leq 0}E = \ldots \to E^{-1} \to E^0 \to \operatorname{im} d^0 \to \ldots,$$
$$B = \tau_{\leq 0}E = \ldots \to 0 \to \ker d^0 \to E^1 \to \ldots.$$

The pairs (A, u) and (B, u) are unique up to unique isomorphism (because for any object $W \in D^{\leq 0}$ we have $\operatorname{Hom}(W, A) \cong \operatorname{Hom}(W, E)$). We shall write it as

$$\tau_{\leq 0} E \to E \to \tau_{\geq 1} E \xrightarrow{+1} .$$

2.3. heart. —

Théorème 2.4. — *The subcategory* $\operatorname{Core}(D) = D^{\leq 0} \cap D^{\geq 0}$, *called the* **core** *or the* **heart** *of D is an abelian category with the following* ker *and* coker *: for $f : X \to Y$ we take a distinguished triangle* $X \xrightarrow{f} Y \to Z \to X[1]$ *and we have* $\ker f = \tau_{\leq 0}(Z[-1])$ *and* $\operatorname{coker} f = \tau_{\geq 0} Z$.

Démonstration. — See Kiehls–Weissauer or Hotta–Takeuchi–Tanisaki. □

Proposition 2.5. — *For $X, Y, Z \in \operatorname{Core}(D)$, a sequence $0 \to X \to Y \to Z \to 0$ is exact in* $\operatorname{Core}(D)$ *if and only if there is a morphism $Z \to X[1]$ in D which completes it into a distinguished triangle* $X \to Y \to Z \xrightarrow{+1}$.

2.6. Cohomology functor. — Define $\mathrm{H}^0 : D \to \operatorname{Core}(D)$ as $\mathrm{H}^0 = \tau_{\geq 0}\tau_{\leq 0} \cong \tau_{\leq 0}\tau_{\geq 0}$. We denote $\mathrm{H}^n = \mathrm{H}^0 \circ [-n]$ for $n \in \mathbf{Z}$.

Given a distinguished triangle $X \to Y \to Z \to X[1]$ we have a long exact sequence

$$\ldots \to \mathrm{H}^{-1}(Z) \to \mathrm{H}^0(X) \to \mathrm{H}^0(Y) \to \mathrm{H}^0(Z) \to \mathrm{H}^1(Z) \to \ldots$$

3. The derived category $\mathrm{D}^b_c(X, \overline{\mathbf{Q}}_\ell)$

Let X be a quasi-projective scheme over a finite or algebraically closed field. There is a triangulated category $\mathrm{D}^b_C(X, \overline{\mathbf{Q}}_\ell)$ with the following properties :

1. there is a "standard" t-structure $(D^{\leq 0}, D^{\geq 0})$ such that
$$\operatorname{Core}^{\mathrm{std}}\left(\mathrm{D}^b_c(X, \overline{\mathbf{Q}}_\ell)\right) \xrightarrow{\cong} \{\overline{\mathbf{Q}}_\ell - \text{sheaves}\}$$
2. $$\mathrm{D}^b_c(X, \overline{\mathbf{Q}}_\ell) = \varinjlim_{\mathbf{Q}_\ell \subseteq E} \varprojlim_r \mathrm{D}^b_{\mathrm{ctf}}(X, \mathcal{O}_E / \pi^r \mathcal{O}_E) \otimes E$$
3. We can define $f_*, f^*, f_!, f^!, \mathcal{H}\mathrm{om}$ and $\otimes$ for $f : X \to Y$. When f is smooth and equidimensional of dimension d, then $f^! = f^*[2d](d)$. When f is proper, we have $f_! = f_*$.

3.1. Dualizing functor $\mathbf{D}_X$. — Let $\varphi : X \to S = \operatorname{Spec}(k)$. The object $\omega_X = \varphi^! \overline{\mathbf{Q}}_{\ell, S}$ is called the **dualising complex** and $\mathbf{D}_X = \mathcal{H}\mathrm{om}(-, \omega_X)$ is called the **dualising functor**.

It has the following properties :

1. $$[1] \circ \mathbf{D}_X = \mathbf{D}_X \circ [-1]$$
2. Let $f : X \to Y$. Then $f_* \circ \mathbf{D}_X \cong \mathbf{D}_Y \circ f_!$.
3. $\mathrm{id}_X \cong \mathbf{D}_X \circ \mathbf{D}_X$.

4. Perverse sheaves

Théorème 4.1. — *Defining*

$$ {}^{p}D^{\leq 0} = \left\{ B \in \mathrm{D}^{b}_{c}\left(X, \overline{\mathbf{Q}}_{\ell}\right) \;;\; \dim \operatorname{supp} \mathcal{H}^{-i} B \leq i,\ \forall i \in \mathbf{Z} \right\} $$
$$ {}^{p}D^{\geq 0} = \left\{ B \in \mathrm{D}^{b}_{c}\left(X, \overline{\mathbf{Q}}_{\ell}\right) \;;\; \dim \operatorname{supp} \mathcal{H}^{-i} \mathbf{D}_{X} B \leq i,\ \forall i \in \mathbf{Z} \right\} $$

Then it is a t-structure.

We define the category of **perverse sheaves** $\mathrm{Perv}(X) = {}^{p}D^{\leq 0} \cap {}^{p}D^{\geq 0}$ as the core of this t-structure.

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