NOTES OF 14 MARCH 2018 : $\mathcal{D}$-MODULES

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1. Setting

Let X be smooth algebraic variety over $k = \mathbf{C}$. Let $\mathcal{O}_X$ be the structure sheaf (algebraic or holomorphic) of X, Ω^i_X the sheaf of differential i-forms of X and Θ_X the sheaf of vector fields on X.

For all $x \in X$, the fiber of Θ_X at x is the tangent space $T_x X$.

Remarque 1.1. — We have $\Theta_X = \mathcal{H}\mathrm{om}_{\mathcal{O}_X}\left(\Omega^1_X, \mathcal{O}_X\right)$.

We have an embedding $\Theta_X = \mathcal{E}\mathrm{nd}_{\mathbf{C}}(\mathcal{O}_X)$, in terms of Lie derivatives : for ξ a vector field

$$L_\xi f = (x \mapsto df_x(\xi(x)))$$

Via the Lie derivatives, vector fields also act on differential forms.

2. $\mathcal{D}$-modules

Définition 2.1. — Let $\mathcal{D}_X$ be the sub-$\mathbf{C}$-algebra generated by Θ_X and Θ_X inside $\mathcal{E}\mathrm{nd}_{\mathbf{C}}(\mathcal{O}_X)$.

If we have a local coordinate chart $(x_1, \cdots, x_n)$, then any section of $\mathcal{D}_X$ is locally written as

$$\sum_{i_1,\ldots,i_n \geq 0} a_{i_1,\ldots,i_n} \partial_1^{i_1} \cdots \partial_n^{i_n}$$

We define

$$\mathcal{M}_l(\mathcal{D}_X) = \text{category of left } \mathcal{D}_X\text{-modules}$$
$$\mathcal{M}_r(\mathcal{D}_X) = \text{category of right } \mathcal{D}_X\text{-modules.}$$

Elements of $\mathcal{D}_X$ can be thought of as functions on T^*X. The sheaf of ring $\mathcal{D}_X$ has a filtration $\mathcal{D}^0 \subseteq \mathcal{D}^1 \subseteq \mathcal{D}^2 \subseteq \cdots$ with

$$\mathcal{D}^0 = \mathcal{O}_X, \quad \mathcal{D}^1 = \mathcal{O}_X \oplus \Theta_X, \quad \mathcal{D}^i \mathcal{D}^j \subseteq \mathcal{D}^{i+j}$$

Let $\mathrm{gr}\,\mathcal{D}_X$ be the graded ring. We have $\mathrm{gr}\,\mathcal{D}_X \cong \mathrm{S}(\Theta_x) \hookrightarrow \mathcal{O}_{T^*X}$.

3. Functorialities

We define the functors of pull-back and push-forward of $\mathcal{D}_X$-modules.

Let $f : X \to Y$ be an embedding of smooth varieties. There are functors

$$\mathcal{M}_l(\mathcal{D}_X) \ni f^*N \longleftarrow\!\!\dashv N \in \mathcal{M}_l(\mathcal{D}_Y)$$

$$\mathcal{M}_l(\mathcal{D}_X) \ni M \longmapsto f_*M \in \mathcal{M}_l(\mathcal{D}_Y)$$

We have the usual pull-back of sheaves $f^{-1}N$. How to upgrade it to a $\mathcal{D}_X$-module?

In the case of $\mathcal{O}$-modules, we put

$$f^*N = O_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}N$$

To define the action of Θ_X, consider

$$\widetilde{f} : TX \to X \times TY$$
$$v \mapsto (v, df_x(v))$$

Let $\xi \in \Theta_X$. To express the push-forward of ξ along $\widetilde{f}$, we choose a coordinate system $\{y_i\}$ so that $\xi = \sum_i a(x) \otimes \partial_{y_i}$. It is then natural to define

$$\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y.$$

Then we can define the pull-back of $\mathcal{D}$-modules

$$f^*N = \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}N$$

The push-forward is more complicated. We shall first talk about the local construction.

Let $i : X \hookrightarrow Y$ be an embedding. We can choose coordinates on Y, say $(y_1, \cdots, y_m)$ in such a way that $X = \{y_{n+1} = \cdots = y_m = 0\}$.

For any right $\mathcal{D}_X$-module M, we let

$$i_*M = \bigoplus_{\alpha_{n+1},\ldots,\alpha_m} M\partial_{n+1}^{\alpha_{n+1}} \cdots \partial_m^{\alpha_m}.$$

We define in general for any right $\mathcal{D}_X$-module M the direct image

$${}^{\mathcal{D}}f_*M = {}^{\mathcal{O}}f_*M \otimes_{f_*\mathcal{D}_X} \mathcal{D}_Y,$$

which is a right $\mathcal{D}_Y$-module.

The push-forward for left modules are more complicated, but in the local chart, we have

$$i_*M = \bigoplus_{\alpha_{n+1},\ldots,\alpha_m} \partial_{n+1}^{\alpha_{n+1}} \cdots \partial_m^{\alpha_m} M.$$

For example, consider the embedding $\{0\} \hookrightarrow \mathbf{A}^1$. Then $\Gamma(\mathbf{A}^1, \mathcal{D}_{\mathbf{A}^1}) = \mathbf{C}\langle x, \partial\rangle / ([\partial, x] = 1)$. We have

$$i_*\mathbf{C} = \mathcal{D}_{\mathbf{A}^1}/(x) = i_*\mathbf{C} = \bigoplus_{i\geq 0} \partial^i \mathbf{C} = \bigoplus_{i\geq 0} \mathbf{C}v_i$$

with the action of $\mathcal{D}_{\mathbf{A}^1}$ defined by

$$xv_0 = 0, \quad \partial v_i = v_{i+1}, \quad xv_i = -iv_{i-1}.$$

4. Theorem of Kashiwara

Let $Z \subseteq Y$ be a closed subvariety of Y. We denote I_Z the defining ideal. We define

$$\mathcal{M}_Z(Y) \subseteq \mathcal{M}_l(Y)$$

be the full subcategory of left $\mathcal{D}_Y$-modules M on which I_Z acts nilpotently.

***Théorème 4.1* (Kashiwara).** — *Let $X \hookrightarrow Y$ be a closed embedding of smooth varieties. Then i_* induces an equivalence of category*

$$i_* : \mathcal{M}(X) \xrightarrow{\cong} \mathcal{M}_X(Y).$$

5. More general definition of push-forwards

Let us recall the derived categories.

Consider the de Rham complex

$$0 \to \mathcal{O}_X \to \Omega^1_X \to \ldots \to \Omega^{d_X}_X \to 0$$

We define for any $M \in \mathcal{M}(X)$,

$$\mathrm{DR}_X(M) = 0 \to \Omega^0_X \otimes_{\mathcal{O}_X} M \to \Omega^1_X \otimes_{\mathcal{O}_X} M \to \ldots \to \Omega^{d_X}_X \otimes_{\mathcal{O}_X} M \to 0,$$

with the differential

$$\nabla : M \to \Omega^1_X \otimes_{\mathcal{O}_X} M$$
$$m \mapsto \sum_i dx_i \otimes \partial_i M$$

In particular, $\mathrm{DR}_X(\mathcal{D}_X)$ is

$$0 \to \Omega^0_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \ldots \to \Omega^{d_X}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \to 0$$

which is a locally free resolution of $\Omega^{d_X}_X$.

Thus

$$\mathrm{DR}_X(M) = \mathrm{DR}_X(\mathcal{D}_X) \otimes_{\mathcal{D}_X} M$$

represent the derived tensor product $\Omega_X \otimes^{\mathrm{L}}_{\mathcal{D}_X} M$ in the derived category $\mathrm{D}^+(\mathcal{M}(X))$ thus we may define

$$f_* M = \Omega^{\otimes -1}_Y \otimes^{\mathrm{L}}_{\mathcal{O}_Y} \mathrm{R}f_* \left(\left(\Omega_X \otimes^{\mathrm{L}}_{\mathcal{O}_X} M \right) \otimes_{\mathcal{D}_X} \mathcal{D}_{X \to Y} \right).$$

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