NOTES OF 18 AVRIL 2018 : PERVERSE SHEAVES–DECOMPOSITION THEOREM

Hironori Oya

1. Perverse sheaves

Let X be a quasi-projective scheme over a field k. Recall that

$$ {}^p\mathrm{D}^{\leq 0}(X) = \left\{ B \in \mathrm{D}^b_c\left(X, \overline{\mathbf{Q}}_\ell\right) \; ; \; \dim \operatorname{supp} \mathcal{H}^{-1} B \leq i, \; \forall i \in \mathbf{Z} \right\} $$
$$ {}^p\mathrm{D}^{\geq 0}(X) = \left\{ B \in \mathrm{D}^b_c\left(X, \overline{\mathbf{Q}}_\ell\right) \; ; \; \dim \operatorname{supp} \mathcal{H}^{-1} \mathbf{D}_X B \leq i, \; \forall i \in \mathbf{Z} \right\} $$

is a t-structure and that $\operatorname{Perv}\left(X, \overline{\mathbf{Q}}_\ell\right) = {}^p\mathrm{D}^{\leq 0}(X) \cap {}^p\mathrm{D}^{\geq 0}(X)$ is defined to be the category of perverse sheaves on X. Here $\mathbf{D}_X = \mathcal{H}\mathrm{om}\left(-, \omega_X\right)$ is the dualising functor so that $\mathrm{id} \cong \mathbf{D}_X \circ \mathbf{D}_X$ and that $f_* \circ \mathbf{D}_X \cong \mathbf{D}_Y \circ f_!$ for any $f : X \to Y$.

If $\varphi : X \to S = \operatorname{Spec} \mathbf{k}$ denotes the projection, then $\omega_X = \varphi^! \left(\overline{\mathbf{Q}}_\ell, S\right)$.

Remarque 1.1. — Let $\mathcal{F}$ be a $\overline{\mathbf{Q}}_\ell$-sheaf. The support of $\mathcal{E}$ is defined to be

$$ \operatorname{supp} \mathcal{F} = \overline{\left\{ j\left(\overline{x}\right) \in X \; ; \; j : \overline{x} \to X \text{ is a geometric point, } j^* \mathcal{F} \neq 0 \right\}} $$

A geometric point is defined to be a morphism $j : \operatorname{Spec} K \to X$ where K is a separably closed field.

The category $\operatorname{Perv}(X)$ is preserved by Verdier duality since $\mathbf{D}_X$ exchanges ${}^p\mathrm{D}^{\leq 0}$ and ${}^p\mathrm{D}^{\geq 0}$.

Remarque 1.2. — If X is smooth and equi-dimensional, then $\omega_X = \overline{\mathbf{Q}}_\ell \left[2 \dim X\right] \left(\dim X\right)$. More generally, if $B \in \operatorname{Perv}(X)$ is a smooth complex (local system), then the cohomological degree is concentrated on $-\dim X$.

2. Gluing

Let $j : U \hookrightarrow X$ be an open inclusion and $i : Y \hookrightarrow X$ its complement. Then

$$ i_* : \operatorname{Perv}(Y) \to \operatorname{Perv}(X) $$
$$ i^* : {}^p\mathrm{D}^{\geq 0}(X) \to {}^p\mathrm{D}^{\geq 0}(Y) $$
$$ i^! : {}^p\mathrm{D}^{\leq 0}(X) \to {}^p\mathrm{D}^{\leq 0}(Y) $$
$$ j^* : \operatorname{Perv}(X) \to \operatorname{Perv}(U) $$
$$ j_* : {}^p\mathrm{D}^{\leq 0}(U) \to {}^p\mathrm{D}^{\leq 0}(X) $$
$$ j_! : {}^p\mathrm{D}^{\geq 0}(U) \to {}^p\mathrm{D}^{\geq 0}(X) $$

One can compose them with the perverse cohomology functor ${}^pH^0$ to get perverse sheaves. We denote the composites by ${}^p i_*, {}^p i^*$, etc.

Besides, there is a unique functor $j_{!*} : \mathrm{Perv}(U) \to \mathrm{Perv}(X)$, called **intermediate extension**, such that

$$j^* \circ j_{!*} \cong \mathrm{id}, \quad i^* \circ j_{!*} \cong 0, \quad i^! \circ j_{!*} \cong 0.$$

Concretely, it is decribed by $\mathrm{image}\left({}^p j_! \to {}^p j_*\right)$. It has the property that $\mathbf{D}_X \circ j_{!*} \cong j_{!*} \circ \mathbf{D}_U$. If U is smooth, then $\mathbf{D}_X\left(j_{!*}\overline{\mathbf{Q}}_\ell[\dim U]\right) \cong j_{!*}\overline{\mathbf{Q}}_\ell[\dim U](\dim U)$. Thus $\overline{\mathbf{Q}}_\ell[\dim U]$ is self-dual up to Tate twist.

Remarque 2.1. — In representation theory, quite often a canonical basis is obtained as intermediate extension of some self-dual sheaf, which is again self-dual. This "categorifies" the bar-invariance of canonical basis. Moreover, the PBW-basis is often the ${}^p j_!$ extension.

Théorème 2.2 **(classification of simple objects in** $\mathrm{Perv}(X)$**)**. — *Let $B \in \mathrm{Perv}(X)$ is an object. Then B is simple if and only if there is an irreducible smooth locally open $U \xrightarrow{j\ open} Y \xrightarrow{i\ closed} X$ and a smooth irreducible $\overline{\mathbf{Q}}_\ell$-sheaf $\mathcal{F}$ on U and an isomorphism*

$$B \cong i_* j_{!*}(\mathcal{F}[\dim U]) =: \mathrm{IC}(\mathcal{F}).$$

Remarque 2.3. — The category $\mathrm{Perv}(X)$ is noetherian and artinian.

3. Decomposition theorem of BBDG

Let X_0 be an separated scheme of finite type over a finite field $\mathbf{k}$. We fix an isomorphism $\tau : \overline{\mathbf{Q}}_\ell \cong \mathbf{C}$.

We say that $B_0 \in \mathrm{D}_c^b\left(X_0, \overline{\mathbf{Q}}_\ell\right)$ is **τ-mixed** all its cohomology sheaves are τ-mixed. A $\overline{\mathbf{Q}}_\ell$-sheaf is τ-mixed if it admits a filtration whose succesive quotient is τ-pure. A sheaf is τ-pure of weight $w \in \mathbf{R}$ if for each closed point $x \in |X_0|$, the Frobenius $\mathrm{F}_x : \mathcal{F}_{\overline{x}} \to \mathcal{F}_{\overline{x}}$ has its eigenvalues α satisfying $|\tau(\alpha)| = \# k(x)^{w/2}$.

For any τ-mixed sheaf $\mathcal{F}_0$, let

$$w(\mathcal{F}_0) = \max\left\{\tau\text{-weights appearing in the successive quotients}\right\}.$$

For any τ-mixed complex $\mathcal{B}_0$, let

$$w(\mathcal{F}_0) = \max\left\{w\left(\mathcal{H}^i(B)\right) - i\right\}.$$

We say that B_0 is τ-pure of weight β if

$$w(B_0) = -w\left(\mathbf{D}_{X_0}(B_0)\right) = \beta$$

Théorème 3.1 **(Decompoisition theorem)**. — *Let $B_0 \in \mathrm{D}_c^b\left(X_0, \overline{\mathbf{Q}}_\ell\right)$ be τ-pure and let B be the base change of B_0 on $X = X_0 \otimes_k \overline{k}$. Then*

$$B \cong \bigoplus_{\substack{A \in \mathrm{Perv}(X)\ simple \\ i \in \mathbf{Z}}} A[i]^{\oplus m_{A,i}}$$

Corollaire 3.2. — *Let $f_0 : X_0 \to Y_0$ be a proper morphism and $U_0 \subseteq X_0$ a smooth locally closed. Then $f_* \mathrm{IC}\left(U, \overline{\mathbf{Q}}_\ell\right)$ is τ-pure and admits a decomposition into a sum of simple perverse sheaves on $Y = Y_0 \otimes_k \overline{k}$.*

HIRONORI OYA