NOTES OF 14 MARCH 2018 : $\mathcal{D}$-MODULES : HORSING AROUND

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1. Plan

In this talk, we will talk about

1. $\mathcal{D}$-algebras
2. Lie algebroids
3. Equivariant $\mathcal{D}$-modules
4. Twisted differential operators
5. Monodromic $\mathcal{D}$-modules

2. $\mathcal{D}$-algebras

Recall that for a smooth variety X over $k = \mathbf{C}$ we have defined $\mathcal{D}_X$ as the subring generated by $\mathcal{O}_X$ and $\mathcal{T}_X = \mathcal{D}\mathrm{er}\,\mathcal{O}_X$ inside the sheaf of ring $\mathcal{E}\mathrm{nd}_{\mathbf{C}}\,\mathcal{O}_X$.

For example, for $X = \operatorname{Spec} k[x] = \mathbf{A}^1$ we have $\mathcal{D}_{\mathbf{A}^1} = k[x,\partial]/\partial x - x\partial = 1$, with $\deg \partial = 1$ and $\deg x = \deg 1 = 0$.

Roughly, a $\mathcal{D}$-algebra is a sort of $\mathcal{O}_X$-algebra (not actually) with a exhausting filtration $\mathcal{O} \subseteq \mathcal{D}^1 \subseteq \mathcal{D}^2 \subseteq \cdots$ such that $\forall \xi \in \mathcal{D}^i$ and $\forall f \in \mathcal{O}_X$, we have $[f,\xi] \in \mathcal{D}^{i-1}$.

Définition 2.1. — A differential bimodule over a commutative k-algebra R is a $R \otimes R$-module M such that $M = \bigcup M_i^\vee$, where $M_i^\vee = \left\{ m \in M; I^{i+1}m = 0 \right\}$ and $I = \ker(R \otimes R \to R)$.

Remarque 2.2. — $I = \langle f \otimes 1 - 1 \otimes f; f \in R \rangle$.

Définition 2.3. — Forr any X, a differential bimodule over X is a quasi-coherent sheaf over $\mathcal{O}_{X \times X}$, supported at the diagonal.

For example, $X = \mathbf{A}^1$. $\Delta X \subseteq X \times X = \operatorname{Spec} \mathbf{C}[x,\xi]$ with $\Delta X = V(\xi)$. Quasi-coherent sheaves on $X \times X$ are $\mathbf{C}[x,\xi]$-modules. Being supported on diagonal means the sheaf M is annihilated by some power of ξ.

Définition 2.4. — A $\mathcal{D}$-algebra on X is a sheaf of associated algebras $\mathcal{A}$ such that $\mathcal{A}$ is a differential bimodule. More precisely, $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_{X \times X}$-module supported on ΔX which is equipped with a $\mathcal{O}_X$-linear multiplication which makes it into an associated algebra.

For example, take $M, N \in \mathrm{Coh}\, X$, then $\mathrm{Hom}_{\mathbf{C}}(M,N)$ is a $\mathcal{O}_X$-bimodule. If we pick out the sections supported on the diagonal, we get a differential bimodule $\mathrm{Diff}(M,N)$. If $M = N$, we denote $\mathcal{D}_M = \mathrm{Diff}(M,M)$.

3. Lie algebroids

Let G be a semisimple algebraic group and $\mathfrak{g}$ its Lie algebra, we have $\mathrm{U}(\mathfrak{g})$.

If we replace groups with groupoids (stacks), then the concept of Lie algebras should become Lie algebroids and $\mathcal{U}(\mathfrak{g})$ should become $\mathcal{D}$-algebras.

Définition 3.1. — A Lie algebroid L on X is an $\mathcal{O}_X$-module equipped with an "anchor" $\sigma : L \to \mathcal{T}_X$ and a Lie bracket $[,] : L \otimes_{\mathbf{C}} L \to L$ such that

1. $[,]$ is a Lie bracket and σ commutes with Lie brackets.
2. If $l_1, l_2 \in L$ and if $f \in \mathcal{O}_X$, then
$$[l_1, fl_2] = f[l_1, l_2] + \sigma(l_1)(f)L_2$$

Remarque 3.2. — The condition 2 should be viewed as a sort of Leibniz rule.

For example, let $f : F \to X$ be a G-torsor. Define $\widetilde{\mathcal{T}}_f = \{G\text{-invariant vector fields on } F \text{ which lift a vector field on } X\}$. There is a natural morphism $\widetilde{\mathcal{T}}_f \to \mathcal{T}_X$, forgetting the vertical tangent direction, which makes it into a Lie algebroid.

Définition 3.3. — A connection on L is an $\mathcal{O}_X$-module morphism $\mathcal{T}_X \xrightarrow{\nabla} L$ such that $\sigma\nabla = \mathrm{id}$. It is flat if it commutes with brackets, i.e. the curvature morphism $C(\nabla) \in \mathrm{Hom}_{\mathcal{O}_X}\left(\bigwedge^2 T_X, \ker\sigma\right)$ defined by
$$C(\nabla)(\tau_1 \wedge \tau_2) = [\nabla\tau_1, \nabla\tau_2] - \nabla[\tau_1, \tau_2]$$
vanishes.

Let G be acting on X. Then we have an infinitesimal action $\mathfrak{g}_X = \mathfrak{g} \otimes \mathcal{O}_X \xrightarrow{\sigma} \mathcal{T}_X$, which makes $\mathfrak{g}_X$ a Lie algebroid. If $X \to Z$ is a G-torsor, we get an exact sequence
$$0 \to \widetilde{\mathfrak{g}}_X \to \widetilde{\mathcal{T}}_X \to \mathcal{T}_Z \to 0$$

If we have a "groupoid action" on X (see Beĭlinson–Bernstein for details) , then we get a Lie algebroid on X.

Let $\mathcal{A}$ be a $\mathcal{D}$-algebra with $i : \mathcal{O}_X \to \mathcal{A}$. We view $\mathcal{A}$ as left $\mathcal{O}_X$ by left multiplication through i. Then $\mathrm{Lie}\,\mathcal{A} = \{(\tau, a) \in \mathcal{T}_X \times \mathcal{A}; i\tau(f) - i(f)a\}$. There is a canonical morphism $\mathrm{Lie}\,\mathcal{A} \xrightarrow{\sigma} \mathcal{T}_X$ which takes the tangent component. The functor $\mathcal{A} \mapsto \mathrm{Lie}\,\mathcal{A}$ admits a left adjoint $\mathcal{U}$: (Lie algebroids) → ($\mathcal{D}$-algebras). The $\mathcal{D}$-algebra $\mathcal{U}(L)$ is the "smallest" algebra with the following data

1. morphisms $i : \mathcal{O}_X \to \mathcal{U}(L)$ and $i_L : L \to \mathcal{U}(L)$ of let $\mathcal{O}_X$-modules.
2. satisfying
$$i_L(fl) = i(f)i_L(l)$$
and
$$[i_L(l), i(f)] = i(\sigma(l))f$$
.

4. Twisted differential operators

Idea : $\mathcal{D}_X$ is generated by $\mathcal{O}_X$ and $\mathcal{T}_X$ inside $\mathcal{E}\mathrm{nd}_{\mathbf{C}}(\mathcal{O}_X)$. Twisted differential operators (tdo) will be something generated by $\mathcal{O}_X$ and $\mathcal{T}_X$, with some slightly different relations between them.

Définition 4.1. — A tdo is a $\mathcal{D}$-algebra $\mathcal{A}$ (with filtration $\mathcal{A}_0 \subseteq \mathcal{A}_i \subseteq \cdots$) such that

1. $\mathcal{O}_X \to \mathcal{A}_0$ is an isomorphism
2. $\sigma : \mathcal{A}_1/\mathcal{A}_0 \to \mathcal{D}\mathrm{er}\,\mathcal{A}_0 = \mathcal{T}_X$ defined by $\sigma(\partial)(f) = \partial f - f\partial$ is an isomorphism.

Remarque 4.2. — Under these conditions, we have $\mathrm{gr}\,\mathcal{A} \cong \mathrm{Sym}\,\mathcal{T}_X$.

Définition 4.3. — A Picard (Lie) algebroid is a Lie algebroid $\widetilde{\mathcal{T}}$ equipped with a section

$$\mathcal{O}_X \to \widetilde{\mathcal{T}}^0 := \ker \sigma.$$

such that

$$0 \to \mathcal{O}_X \to \widetilde{\mathcal{T}} \xrightarrow{\sigma} \mathcal{T}_X \to 0$$

is exact. i.e. a central extension of $\mathcal{T}_X$ by $\mathcal{O}_X$.

Lemme 4.4. — *There is an equivalence*

$$\begin{aligned} \mathrm{tdo}(X) &\cong \mathrm{Picalgbd}(X) \\ D &\mapsto \mathrm{Lie}\,D \\ \mathcal{U}(\widetilde{T})/(i - 1_{\widetilde{T}}) &\leftarrow \widetilde{T} \end{aligned}$$

Another way to think about it is via twisted cotangent bundles. i.e. $\mathcal{T}_X^*$ - torsor on X, $\psi \xrightarrow{\pi_\psi} X$ with symplectic form such that the projection is a lagrangian fibration, compatible to the $\mathcal{T}_X^*$-action.

Remarque 4.5. — $\mathcal{D}_X$-action on a vector bundle $\mathcal{E} \to X$ is an $\mathcal{O}_X$-morphism $\mathcal{T}_X \to \widetilde{\mathcal{T}}_{\mathcal{E}}$, which is a connection on $\widetilde{\mathcal{T}}_{\mathcal{E}}$ (or on $\mathcal{E}$).

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