NOTES OF 25 AVRIL 2018 : JANTZEN CONJECTURES AND MONODROMIC $\mathcal{D}$-MODULES

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1. Goal

The goals of the talk today is to

(i). give an overview of the Jantzen conjectures and the tools which are needed
(ii). explain the monodromic construction of twisted differential algebras

We follow Beĭlinson–Bernstein.

2. Jantzen conjectures

For a complete survey, see Humphrey, *Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$*, 5.3 and 8.12.

2.1. contravariant forms. — Let $\mathfrak{g}$ be a semi-simple complex Lie algebra, let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be a Cartan decomposition, Δ^+ be the positive roots and $\Pi \subseteq \Delta^+$ be the simple roots. For each $\alpha \in \Delta^+$, we choose an $\mathfrak{sl}_2$-triplet $(x_\alpha, h_\alpha, y_\alpha)$ with $x_i \in \mathfrak{g}_\alpha,\ y_i \in \mathfrak{g}_{-\alpha},\ h_i \in \mathfrak{h}$. We denote $\tau : \mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g})^{\mathrm{op}}$ the anti-automorphism of $\mathrm{U}(\mathfrak{g})$, called **transpose**, which interchanges x_α, y_α for each $\alpha \in \Delta^+$ and fixes $\mathfrak{h}$.

We denote $\mathrm{Mod}(\mathfrak{g}) = \mathrm{Mod}(\mathrm{U}(\mathfrak{g}))$. Let $M \in \mathrm{Mod}(\mathrm{U}(\mathfrak{g}))$.

Définition 2.2. — A symmetric $\mathbf{C}$-bilinear form $(-,-)$ on M is called **contravariant** if

$$\left(xv, v'\right) = \left(v, \tau(x)v'\right), \quad \forall x \in \mathrm{U}(\mathfrak{g}),\ \forall v, v' \in M.$$

Proposition 2.3. — *Let $M \in \mathcal{O}(\mathfrak{g})$ be a highest weight module with highest weight vector $v_0 \in M$.*

(i). There exists a unique contravariant form $(-,-)$ on M normalised such that $(v_0, v_0) = 1$.
(ii). For $\lambda, \mu \in \mathfrak{h}^*$, $(M_\lambda, M_\mu) = 0$ whenever $\mu \neq \lambda$.
(iii). The radical of $(-,-)$ is equal to the unique maximal proper submodule of M.

Those statements are more or less obvious except possibly for the existence. Let

$$\varphi : \mathrm{U}(\mathfrak{g}) = \mathrm{U}(\mathfrak{n}^-) \otimes \mathrm{U}(\mathfrak{h}) \otimes \mathrm{U}(\mathfrak{n}^+) \to \mathrm{U}(\mathfrak{h})$$

be the linear projection which sends the elements in the PBW-basis with x_α or y_α involved to 0. We define a $\mathrm{U}(\mathfrak{h})$-valued symmetric $\mathbf{C}$-bilinear form on $\mathrm{U}(\mathfrak{g})$ by

$$\mathrm{C}\left(u, u'\right) = \varphi\left(\tau(u)u'\right).$$

This is a contravariant form on the regular representation $\mathrm{U}(\mathfrak{g}) \in \mathrm{Mod}(\mathfrak{g})$. For each $\lambda \in \mathfrak{h}^* \cong \mathrm{Spec}\,\mathrm{U}(\mathfrak{h})$, we compose the form C with $\lambda : \mathrm{U}(\mathfrak{h}) \to \mathbf{C}$ to get a $\mathbf{C}$-valued contravariant bilinear form C^λ.

If M is a highest weight module with highest weight vector $v_0 \in M_\lambda$, then we pose

$$\left(uv_0, u'v_0\right) = \mathrm{C}^\lambda\left(u, u\right)$$

for each $u, u' \in \mathfrak{U}(\mathfrak{n}^-)$. It is easy to verify that it defines a contravariant form on M.

2.4. Jantzen filtration. — Let $\lambda \in \mathfrak{h}^*$ and let $M(\lambda)$ be the Verma module of highest weight λ.

Théorème 2.5 **(Jantzen)**. — *There is a filtration by submodules* $M(\lambda) = M(\lambda)^0 \supseteq M(\lambda)^1 \supseteq \ldots$, *with* $M(\lambda)^k = 0$ *for* $k \gg 0$, *such that*

(i). each $M(\lambda)^i / M(\lambda)^{i+1}$ *admits a non-degenerate contravariant form.*
(ii). $M(\lambda)^1$ *is the unique maximal proper submodule.*
(iii).

$$\sum_{i>0} \mathrm{ch}\, M(\lambda)^i = \sum_{\alpha \in \Delta^+, s_\alpha \cdot \lambda < \lambda} \mathrm{ch}\, M(s_\alpha \cdot \lambda)$$

Here $s_\alpha \cdot \lambda = s_\alpha(\lambda + \rho) - \rho = \lambda - \langle \lambda + \rho, \alpha^\vee \rangle \alpha$, so the condition $s_\alpha \cdot \lambda < \lambda$ is equivalent to that $\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbf{N}^*$. We denote

$$\Delta_\lambda = \{\alpha \in \Delta \; ; \; \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbf{Z}\}$$

Démonstration. — The construction goes roughly as follows :

(1). Arguing on the central characters, we know that if $\lambda + \rho \in \mathfrak{h}^*$ is anti-dominant and regular, meaning that $\langle \lambda + \rho, \alpha \rangle \notin \mathbf{N}$, then $M(\lambda)$ is irreducible.
(2). Extending the base field from $\mathbf{C}$ to $K = \mathbf{C}(T)$, we form the Lie algebra $\mathfrak{g}_K = \mathfrak{g} \otimes_{\mathbf{C}} K$, the Cartan $\mathfrak{h}_K = \mathfrak{h} \otimes_{\mathbf{C}} K$ etc. Consider the weight $\lambda_T = \lambda + T\rho \in \mathfrak{h}^*$. Then $\lambda + T\rho + \rho$ is obviously anti-dominant and regular, thus $M(\lambda_T) \in \mathcal{O}(\mathfrak{g}_K)$ is irreducible. The contravariant form on $M(\lambda + T\rho)$ is non-degenerate.
(3). Let $A = \mathbf{C}[T] \subseteq K$. We define $\mathfrak{g}_A = \mathfrak{g} \otimes_{\mathbf{C}} A$, $\mathrm{U}(\mathfrak{g}_A) = \mathrm{U}(\mathfrak{g}) \otimes_{\mathbf{C}} A$, etc. We also have an A-valued contravariant form C^{λ_T} on $\mathrm{U}(\mathfrak{g}_A)$, which induces an A-valued non-degenerate contravariant form on

$$M_A(\lambda_T) = \mathrm{U}(\mathfrak{g}_A) \otimes_{\mathrm{U}(\mathfrak{b}_A^+)} A_{\lambda_T}$$

where $A_{\lambda_T} \in \mathrm{Mod}(\mathfrak{b}_A^+)$ is a free A-module of rank 1 on which $\mathfrak{n}_A^+$ acts by 0 and $\mathfrak{h}_A$ acts by $\lambda_T \in \mathrm{Spec}\,\mathrm{U}(\mathfrak{h}_A)$.

(4). The restriction of the contravariant form induces a non-degenerate A-bilinear form, $(-,-)$ on each weight space $M_{\lambda_T-\nu} := M_A(\lambda_T)_{\lambda_T-\nu}$. For each $i \in \mathbf{N}$, we put

$$M^i_{\lambda_T-\nu} = \left\{v \in M_{\lambda_T-\nu} \; ; \; (v, M_{\lambda_T-\nu}) \subseteq T^i A\right\}.$$

Then

$$M_A(\lambda_T)^i = \sum_{\nu \in \Lambda^{\geq 0}} M^i_{\lambda_T-\nu}$$

is a decreasing filtration by submodules. The Jantzen filtration is defined as the image of this filtration under the quotient $M_A(\lambda_T)/TM_A(\lambda_T) \cong M(\lambda)$.

(5). The contravariant form on each successive quotient $M(\lambda)^i/M(\lambda)^{i+1}$ induced by $T^{-i}(-,-)\big|_{M^i_{\lambda_T-\nu}}$ is a non-degenerate $\mathbf{C}$-valued contravariant form.

The proof of (iii) requires so elementary calculations, which we don't include here. □

2.6. Jantzen conjectures. — It is not *a priori* clear how canonical this construction is. One asks the following questions

(i). Instead of using ρ, we could have defined $\lambda_T = \lambda + T\mu$ for any μ regular. Does the resulting filtration depends on μ ?

(ii). In light of the (ii) of the theorem, is $M(\lambda)^{i+1} = \operatorname{cosoc}\left(\mathrm{M}(\lambda)^{\mathrm{i}}\right)$ true in general, where cosoc means the intersection of maximal proper submodules ?

(iii). If $M(w \cdot \lambda) \subseteq M(\lambda)$, how are the Jantzen filtrations of both related ?

These are called **Jantzen conjectures**, answered by Beĭlinson–Bernstein.

3. Geometric interpretation of Jantzen filtration

See Beĭlinson–Bernstein section 4. One of the main idea of Beĭlinson–Bernstein is to interpret the Jantzen filtration as the monodromy filtration on certain perverse sheaves.

Let $B \subseteq G$ be a Borel subgroup, $N \subseteq B$ be its unipotent radical, $X = G/B$ be the flag variety, $\widetilde{X} = G/N$ be the enhanced flag variety, $H = B/N$ be the quotient torus and W be the Weyl group. For each $w \in W$, let $j : X_w \subseteq X$ be the Schubert cell and let $\lambda \in \mathfrak{h}^*_{\mathbf{Q}}$ be a rational dominant regular weight and let $\chi : \mathrm{Z}(\mathfrak{g}) \to \mathbf{C}$ be the central character induced by λ. Here is a comparison table

representation of $\mathfrak{g}$	$\mathcal{D}$-module	ℓ-adic perverse sheaf
$\operatorname{Mod}(\mathfrak{g})_\chi$	$\operatorname{Mod}\left(\mathcal{D}^\chi_X\right)$	$\operatorname{Perv}\left(\widetilde{X}, \overline{\mathbf{Q}}_\ell\right)_\chi$
$\operatorname{Mod}(\mathfrak{g}, N)$	$\operatorname{Mod}_N\left(\mathcal{D}^\chi_X\right)$	$\operatorname{Perv}_N\left(\widetilde{X}, \overline{\mathbf{Q}}_\ell\right)_\chi$
$M(w_0 w \cdot \lambda)$	$j_!\left(\mathcal{L}_{\lambda, X_w} \otimes_{\mathbf{C}} \mathcal{O}_{\widetilde{X}_w}\right)$	$j_!\mathcal{L}_{\lambda, \widetilde{X}_w}[\dim \widetilde{X}_w]$
$L(w_0 w \cdot \lambda)$	$j_{!*}\left(\mathcal{L}_{\lambda, X_w} \otimes_{\mathbf{C}} \mathcal{O}_{\widetilde{X}_w}\right)$	$\mathrm{IC}\left(\widetilde{X}_w, \mathcal{L}_\lambda\right)$
deformation $\lambda_T = \lambda + T\varphi$	morphism $f_\varphi : \overline{X}_w \to \mathbf{A}^1$	–
multiplication by T	(logorithmic) monodromy	–
Jantzen filtration $M(\lambda)^i$	monodromy filtration $J_!$	–
τ	contravariant duality c	–
contravariant form	$j_!\left(V \otimes I^{(n)}_\varphi\right) \to j_*\left(V \otimes I^{(n)}_\varphi\right)$	–

Here $\mathcal{L}_{\lambda,\widetilde{X}_w}$ is the local system on H with finite monodromy and with eigenvalues equal to $\lambda : \mathbf{X}_*(H) \to \mathbf{C}^\times$, induced on $\widetilde{X}$ and restricted to $\widetilde{X}_w$.

3.1. Jacobson–Morosov theorem. — We recall some basic facts about nilpotent operators on an abelian category.

Let V be a finite dimensional vector space over a field of characteristic 0 and let $s \in \mathrm{End}(V)$ be a nilpotent endomorphism.

Théorème 3.2 **(Morosov, Jacobson, Kostant)**. — *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field of characteristic 0 and let $e \in \mathfrak{g}$ be nilpotent. Then there exists $h, f \in \mathfrak{g}$ such that $(e = s, h, f)$ forms an $\mathfrak{sl}_2$-triplet. Moreover, such pairs (h, f) is a principal homogeneous space of $\mathrm{Rad}^{\mathrm{unip}}(C_G(e))$, where G is a simply connected algebraic group with $\mathrm{Lie}\, G = \mathfrak{g}$.*

Therefore, we can find $h, f \in \mathrm{End}(V)$ as in the theorem. For $n \in \mathbf{Z}$, let $V_n = \{v \in V \,;\, h(v) = nv\}$ be the weight space of h. We can define an increasing filtration $\mu_n V = \bigoplus_{k \le n} V_{-k}$. Then

Proposition 3.3. — *(i). The operator $s^n \in \mathrm{End}(V)$ induce for each $n \ge 0$ and isomorphism*

$$s^n : \mathrm{Gr}^\mu_n V \cong \mathrm{Gr}^\mu_{-n} V$$

(ii). The filtration μ is independent of the pair $(h, f) \in \mathrm{End}(V)$ and is characterises by the property (i).

In fact, we can define the filtration μ recursively and (ii) generalises to an arbitrary abelian category.

Now let V be an object in an abelian category $\mathcal{A}$ and let $s \in \mathrm{End}(V)$ be a nilpotent endomorphism. Let $J_{!i}$, $i \in \mathbf{Z}$ be the increasing filtration on $\ker s$ defined by

$$J_{!i} = \begin{cases} \ker s \cap \operatorname{im} s^{-i}, & i \le 0 \\ \ker s, & i \ge 0 \end{cases}$$

and let J_{*i}, $i \in \mathbf{Z}$ be the increasing filtration on $\operatorname{coker} s$ defined by

$$J_{*i} = \begin{cases} \left(\ker s^i + \operatorname{im} s\right) / \operatorname{im} s, & i \ge 0 \\ \operatorname{coker} s, & i \le 0 \end{cases}$$

It is clear that these filtrations agree with the Jacobson–Morosov filtration μ on ker and coker. Taking $\overline{V} = V / \ker s$. Then s induces a nilpotent $\overline{s} \in \mathrm{End}\left(\overline{V}\right)$.

Lemme 3.4. — *For a nilpotnet endomorphism $s \in \mathrm{End}(V)$ with s^n, there is a unique filtration $(V, \mu) \in \mathrm{Fil}\,\mathcal{A}$ satisfying the following properties,*

(i). $s(\mu_n V) \subseteq \mu_{n-2} V, \quad \forall n \in \mathbf{Z}$

(ii). With $\overline{\mu}$ the Jacobson–Morosov filtration for $\overline{s}$, the following sequence in $\mathrm{Fil}\,\mathcal{A}$ is exact and strict

$$0 \to (\ker s, J_!) \to (V, \mu) \to \left(\overline{V}, \overline{\mu}_{\bullet-1}\right) \to 0$$

Moreover, if we replace (ii) by the following *(ii bis). the follwing sequence in* $\mathrm{Fil}\,\mathcal{A}$ *is exact and strict*

$$0 \to \left(\overline{V}, \overline{\mu}_{\bullet+1}\right) \to (V, \mu) \to (\operatorname{coker} s, J_*) \to 0$$

the resulting filtration μ *remains unchanged.*

3.5. monodromy filtration and Jantzen filtration. — Let Y be a smooth variety, $f : Y \to \mathbf{A}^1$ be a regular function. We have

$$\begin{array}{ccccc} Z = f^{-1}(0) & \xhookrightarrow{i} & Y & \xhookleftarrow{j} & f^{-1}\left(\mathbf{A}^1 \smallsetminus \{0\}\right) = U \\ \downarrow f & & \downarrow f & & \downarrow f \\ \{0\} & \hookrightarrow & \mathbf{A}^1 & \hookleftarrow & \mathbf{A}^1 \smallsetminus \{0\} \end{array}$$

We denote $\mathbf{A}^1 = \operatorname{Spec} \mathbf{C}[t]$. For each $n \in \mathbf{N}$, let $I^{(n)} \in \operatorname{Mod}\left(\mathcal{D}_{\mathbf{A}^1 \smallsetminus \{0\}}\right)$ be the holonomic (left) $\mathcal{D}_{\mathbf{A}^1 \smallsetminus \{0\}}$-module defined by

$$I^{(n)} = \left(\mathcal{D}_{\mathbf{A}^1 \smallsetminus \{0\}} \otimes_{\mathbf{C}} \mathbf{C}[s]/s^n\right) / \left(t\partial_t - s\right)$$

In other words, $I^{(n)}$ is a free rank 1 module over $\mathcal{O}_{\mathbf{A}^1 \smallsetminus \{0\}} \otimes \mathbf{C}[s]/s^n$ with a generator t^s, with the $\mathcal{D}$-module structure given by $t\partial_t(t^s) = st^s$. The $\mathcal{D}$-module $I^{(n)}$ is equipped with an $\mathbf{C}[s]/s^n$-action by multiplication.

For any holonomic $M \in \operatorname{Mod}(\mathcal{D}_U)$, we define $f^s M_U^{(n)} = f^* I^{(n)} \otimes_{\mathcal{O}_U} M_U \in \operatorname{Mod}(\mathcal{D}_U[s]/s^n)$. Then

$$f^s M_U = \varprojlim_n f^* I^{(n)} \otimes_{\mathcal{O}_U} M_U \in \operatorname{Mod}(\mathcal{D}_U[[s]]) .$$

For each $a \in \mathbf{N}$, the endomorphism morphism $s^a : f^s M_U \to f^s M_U$ gives by the adjunction

$$\operatorname{Hom}(f^s M_U, f^s M_U) \cong \operatorname{Hom}(j^* j_! f^s M_U, f^s M_U) \cong \operatorname{Hom}(j_! f^s M_U, j_* f^s M_U)$$

a morphism $s^a : j_! f^s M_U \to j_* f^s M_U$. The holonomicity of M_U implies that $\pi_f^a(M_U) := \operatorname{coker} s^a$ is s-torsion and is again a holonomic $\mathcal{D}_U$-module by forgetting s.

We denote $\Psi_f^{\mathrm{un}} = \pi_f^0$ and $\Xi_f = \pi_f^1$, which are respectively *the nearby cycles with unipotent monodromy* and *the maximal extension (or tilting extension)* functors.

Lemme 3.6. — *(i). The functor* $\pi_f^a : \operatorname{Mod}_h(\mathcal{D}_U) \to \operatorname{Mod}_h(\mathcal{D}_Y)$ *is exact*
(ii). $\pi_f^a \circ \mathbf{D}_U \cong \mathbf{D}_Y \circ \pi_f^a$
(iii). The following natural sequences of functors on $\operatorname{Mod}_h(\mathcal{D}_U)$ *are exact*

$$0 \to j_! \to \Xi_f \to \Psi_f^{\mathrm{un}} \to 0$$

$$0 \to \Psi_f^{\mathrm{un}} \xrightarrow{s} \Xi_f \to j_* \to 0$$

with $j_! \cong \ker\left(s : \Xi_f \to \Xi_f\right)$ *and* $j_* = \operatorname{coker}\left(s : \Xi_f \to \Xi_f\right)$.

Applying the Jacobson–Morosov theorem on s and $\pi_f^a(M_U)$, we get a filtration $\mu^{(a)}$, called the monodromy filtration. Thus, on $j_! = \ker s$ and on $j_* = \operatorname{coker} s$, we have the filtrations $J_!$ and J_* defined above. These are the **geometric Jantzen filtrations**.

We will apply this construction to the case where $Y = \overline{\widetilde{X}_w}$ is the closure of some Schubert cell in the enhanced flag variety and on the category of N-equivariant twisted $\mathcal{D}$-modules on X.

4. Monodromic twisted $\mathcal{D}$-modules

Let X be a complex smooth algebraic variety, let H be an algebraic torus and let $\pi : \widetilde{X} \to X$ be an H-torsor.

4.1. tdo defined by line bundle. — Given any H-character $\lambda \in \mathbf{X}^*(H)$, we can construct a line bundle on X by descent :

$$\mathcal{L}_\lambda = \widetilde{X} \times^H \mathbf{C}_\lambda$$

Then this line bundle gives rise to a tdo $\mathcal{D}_X^\lambda = \mathrm{Diff}\left(\mathcal{L}_\lambda, \mathcal{L}_\lambda\right)$, which is by definition the ring of differential operators on $\mathcal{L}_\lambda$.

Another description : consider the Picard algebroid $\widetilde{T}^\lambda$ on X, whose local sections are $\mathbf{G}_m$-invariant vector fields on the $\mathbf{G}_m$-torsor

$$\pi^\lambda : \widetilde{X}^\lambda = \widetilde{X} \times^H \mathbf{C}_\lambda^\times \to X.$$

Then $\mathcal{D}_X^\lambda$ is isomorphic to the tdo $\mathrm{U}\left(\widetilde{T}^\lambda\right) / \left(1 - 1_{\widetilde{T}}\right)$. Furthermore, we have $\left(\pi_*^\lambda \mathcal{D}_{\widetilde{X}^\lambda}\right)^{\mathbf{G}_\mathrm{m}} \cong \mathrm{U}\left(\widetilde{T}^\lambda\right)$, the center of which is $\mathrm{U}\left(\widetilde{T}^\lambda\right)$ is $\mathrm{U}(\mathbf{C}_\lambda)$ and there is an isomorphism

$$\mathcal{D}_X^\lambda \cong \left(\pi_*^\lambda \mathcal{D}_{\widetilde{X}^\lambda}\right)^{\mathbf{G}_\mathrm{m}} \otimes_{\mathrm{U}(\mathbf{C}_\lambda)} \mathrm{U}\left(\mathbf{C}_\lambda\right) / \left(1_{\mathbf{C}_\lambda} - 1\right).$$

Similarly,

$$\mathcal{D}_X^\lambda \cong \left(\pi_* \mathcal{D}_{\widetilde{X}}\right)^H \otimes_{\mathrm{U}(\mathfrak{h})} \mathbf{C}_\lambda$$

Here, the morphism $\mathrm{U}\left(\mathfrak{h}\right) \to \mathbf{C}_\lambda$ is given by $\lambda \in \mathbf{X}^*\left(H\right) \subseteq \mathrm{Spec}\,\mathrm{U}\left(\mathfrak{h}\right)$ regarded as maximal ideal.

More generally, given any weight $\lambda \in \mathfrak{h}^*$, we define

$$\mathcal{D}_X^\lambda = \left(\pi_* \mathcal{D}_{\widetilde{X}}\right)^H \otimes_{\mathrm{U}(\mathfrak{h})} \mathbf{C}_\lambda$$
$$\mathcal{D}_X^{\widetilde{\lambda}} = \left(\pi_* \mathcal{D}_{\widetilde{X}}\right)^H \otimes_{\mathrm{U}(\mathfrak{h})} \widehat{\mathcal{O}}_{\mathfrak{h}^*,\lambda}$$

The algebra $\widetilde{\mathcal{D}}_X = \left(\pi_* \mathcal{D}_{\widetilde{X}}\right)^H$ should therefore be regarded as a $\mathfrak{h}^*$-family of tdos on X with $\mathcal{D}_X^\lambda$ being the fiber at $\lambda \in \mathfrak{h}^*$ and $\mathcal{D}_X^{\lambda^n}$ being the fiber at the infinitesimal neighbourhood of λ.

4.2. monodromic tdo. — We should describe $\mathrm{Mod}\left(\mathcal{D}_X^\lambda\right)$ as certain $\mathcal{D}$-modules on $\widetilde{X}$. As the projection $\pi : \widetilde{X} \to X$ is affine, the direct image

$$\pi_* : \mathrm{Mod}\left(\mathcal{D}_{\widetilde{X}}\right) \to \mathrm{Mod}\left(\pi_* \mathcal{D}_{\widetilde{X}}\right)$$

is an equivalence of category. Let $\mathrm{Mod}_{\mathrm{fin}}\left(\pi_* \mathcal{D}_{\widetilde{X}}\right) \subseteq \mathrm{Mod}\left(\pi_* \mathcal{D}_{\widetilde{X}}\right)$ be the subcategory of modules on which $\mathrm{U}\left(\mathfrak{h}\right) \subseteq \pi_* \mathcal{D}_{\widetilde{X}}$ acts through a finite quotient.

There is a decomposition of category

$$\mathrm{Mod}_{\mathrm{fin}}\left(\pi_* \mathcal{D}_{\widetilde{X}}\right) \cong \bigoplus_{\lambda \in \mathfrak{h}^*} \mathrm{Mod}\left(\mathcal{D}_X^{\widetilde{\lambda}}\right).$$

The inverse functor $(\pi_*)^*$ identify them with some subcategories of $\mathrm{Mod}\left(\mathcal{D}_{\widetilde{X}}\right)$. Those can be described in terms of weak Harish–Chandra modules of the pair $\left(\widetilde{X}, H\right)$.

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