NOTES OF 4 AVRIL 2018 : ℓ-ADIC ÉTALE SHEAVES AND THE THEORY OF WEIGHTS

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1. References

We follow mainly SGA 4 1/2, Freitag–Kiehls and Kiehls–Weissauer.

2. Introduction

2.1. Counting rational points on a variety. — Let X_0 be an algebraic variety over a finite field $\mathbf{F}_q$. One introduces the zeta function for X_0 :

$$\zeta(X_0, s) = \sum_{S \in \mathcal{Z}_0(X_0)_+} \frac{1}{(\# \deg S)^s}$$

where $\mathcal{Z}_0(X_0)_+$ is the semigroup of effective algebraic 0-cycles on X_0. Its Euler product formula takes the form

$$\zeta(X_0, s) = \prod_{x \in |X_0|} \frac{1}{1 - (\# k(x))^{-s}}$$

Here $|X_0|$ is the set of closed points of X_0 and $k(x)$ is the residue field (which is a finite extension of $\mathbf{F}_q$) of x. If we let $\deg(x) = [k(x) : \mathbf{F}_q]$ and we introduce a change of variable $T = q^{-s}$, then

$$\log \zeta(X_0, T) = \sum_{x \in |X_0|} -\log\left(1 - T^{\deg(x)}\right) = \sum_{x \in |X_0|} \sum_{k \geq 1} \frac{T^{k \deg(x)}}{k} = \sum_{n \geq 1} \left(\sum_{d | n} \sum_{\substack{x \in |X_0| \\ \deg(x) = d}} d \right) \frac{T^n}{n}$$

On the other hand, we have the following counting formula (easily derived from the Galois theory) for each $k \geq 1$

$$\# X_0(\mathbf{F}_{q^n}) = \sum_{d | n} \sum_{\substack{x \in |X_0| \\ \deg(x) = d}} d.$$

Therefore the power series

$$\log \zeta(X_0, T) = \sum_{n \geq 1} \# X_0(\mathbf{F}_{q^n}) \frac{T^n}{n}$$

encode the number of rational points of X_0 and its base change to $\mathbf{F}_{q^n}$. Taking the derivative,

$$T \frac{\zeta'(X_0, T)}{\zeta(X_0, T)} = \sum_{n \geq 1} \# X_0(\mathbf{F}_{q^n}) T^n.$$

2.2. Weil conjectures. — For X_0 a projective smooth algebraic variety over $\mathbf{F}_q$, André Weil famously conjectured the following properties on $\zeta(X_0,T)$:

1. $\zeta(X_0,T)$ is a rational function on T, in the forms
$$\zeta(X_0,T)=\prod_{k\geq 0}^{\dim X_0} P_k(T)^{(-1)^{k+1}}$$
where $P_k(T)\in\mathbf{Z}[T]$
2. there is a functional equation between $\zeta(X_0,1/q^{\dim X_0}T)$ and $\zeta(X_0,T)$.
3. The roots of the polynomials $P_k(T)$ are all of absolute value equal to $q^{-k/2}$.

Some heuristics behind 1 and 2 :

– Denote $\mathbf{k}=\overline{\mathbf{F}}_q$ an algebraic closure. Let $X=X_0\otimes_{\mathbf{F}_q}\mathbf{k}$. This should be field as an analogue of algebraic manifold.
– Let $q=p^r$ where p is a prime number. Rational points $X_0(\mathbf{F}_{q^n})$ should be viewed as fixed points the Frobenius endomorphism (which is a morphism of varieties over $\mathbf{k}$)
$$\begin{aligned}\mathrm{F}^n_{X/\mathbf{k}}:X&\to X\\ x&\mapsto x^{q^n}\end{aligned}$$
– there should be an analogue of singular cohomology $\mathrm{H}^k(X)$ and Lefschetz fixed-points formula, stating that
$$\chi\left(\mathrm{F}^n_{X/\mathbf{k}}\right)=\sum_{k\geq 0}^{2\dim X}(-1)^k\,\mathrm{tr}\left(\left(\mathrm{F}^n_{X/\mathbf{k}}\right)^*\ ;\ \mathrm{H}^k\left(X\right)\right)$$
where χ is the Lefschetz index.
– The graph of the Frobenius morphism in $X\times X$ is transversal to the diagonal Δ_X, so that the fixed points being the intersection are multiplicity-free and therefore $\#X_0\left(\mathbf{F}_{q^n}\right)=\chi\left(\mathrm{F}^n_{X/\mathbf{k}}\right)$.
– Putting all n together, we will have
$$T\frac{\zeta'\left(X_0,T\right)}{\zeta\left(X_0,T\right)}=\sum_{k\geq 0}^{2\dim X}(-1)^k\sum_{n\geq 1}\mathrm{tr}\left(\left(\mathrm{F}^n_{X/\mathbf{k}}\right)^*\ ;\ \mathrm{H}^k\left(X\right)\right)T^n$$
and thus
$$\zeta\left(X_0,T\right)=\prod_{k\geq 0}^{2\dim X}\det\left(1-\mathrm{F}^*_{X/\mathbf{k}}T\ ;\ \mathrm{H}^k(X)\right)^{(-1)^{k+1}}.$$
Setting $P_k(T)=\det\left(1-\mathrm{F}^*_{X/\mathbf{k}}T\ ;\ \mathrm{H}^k(X)\right)^{(-1)^{k+1}}$.
– There should be a Poincaré duality, relating $P_k(T)$ and $P_{2n-k}(1/q^{\dim X_0}T)$ thus giving a functional equation between $\zeta(X_0,T)$ and $\zeta(X_0,1/q^{\dim X_0}T)$.

The remaining question 3 is that the roots of $P_k(T)=\det\left(1-\mathrm{F}^*_{X/\mathbf{k}}\ ;\ \mathrm{H}^k(X)\right)$ should have absolute value $q^{-k/2}$. In other words, the absolute value of each eigenvalue of the Frobenius $\mathrm{F}^*_{X/\mathbf{k}}$ acting on the k-th cohomology $\mathrm{H}^k(X)$ should be $q^{k/2}$. We shall call it the **Riemann hypothesis**.

3. Étale topology

In the following, we consider general schemes

Définition 3.1. — A morphism of varieties $f:X\to Y$ is called **étale** if

1. f is locally of finite presentation
2. f is flat
3. $\Omega_{X/Y}=0$.

Let X be a scheme, we define the **small étale site** $\text{Ét}(X)$. $\text{Ét}(X)$ is a category whose objects are étale morphisms $Y \to X$ and whose morphisms are commutative triangles

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

Remarque 3.2. — 1. In this the above diagram, $Y \to Y'$ is automatically étale.
2. Etale morphisms are open mappings on the underlying Zariski topology.

In the following, the structure morphism $Y \to X$ will be omited.

We say that a morphism $Y \to Y'$ in $\text{Ét}(X)$ is an **étale covering** if it is surjective.

3.3. Étale topos. — A presheaf on $\text{Ét}(X)$ is a functor

$$\mathscr{G} : \text{Ét}(X)^{\text{op}} \to \mathbf{Set}\,.$$

A presheaf $\mathscr{G}$ on $\text{Ét}(X)$ is called a sheaf if the following conditions are satisfied

1. If $\{U_i\}_{i\in I}$ is a family of objects in $\text{Ét}(X)$, then the natural mapping
$$\mathscr{G}\left(\bigsqcup_{i\in I} U\right) \to \prod_{i\in I} \mathscr{G}(U_i)$$
is an isomorphism and
2. for every étale covering $g : U \to Y$ in $\text{Ét}(X)$, in the following diagram
$$\mathscr{G}(Y) \xrightarrow{g^*} \mathscr{G}(U) \underset{\text{pr}_2^*}{\overset{\text{pr}_1^*}{\rightrightarrows}} \mathscr{G}(U \times_Y U)$$
g^* is an equaliser of pr_1^* and pr_2^*.

The category of sheaves on $\text{Ét}(X)$ will be denoted $X_{\text{ét}}$ and is called the **étale topos** of X.

The embedding functor $X_{\text{ét}} \to \text{Fct}\left(\text{Ét}(X)^{\text{op}}, \mathbf{Set}\right)$ admits a left adjoint

$$\# : \text{Fct}\left(\text{Ét}(X)^{\text{op}}, \mathbf{Set}\right) \to X_{\text{ét}}$$

For any presheaf $\mathscr{G} \in \text{Fct}\left(\text{Ét}(X)^{\text{op}}, \mathbf{Set}\right)$, the sheaf $\mathscr{G}^{\#}$ is called the **sheaf associated** to $\mathscr{G}$.

3.4. functorialities. — For any morphism of schemes $f : X \to Y$, there is an induced morphism on the small étale sites $f^{-1} : \text{Ét}(Y) \to \text{Ét}(X)$ defined by base change. The functor f^{-1} preserves finite limits :

$$f^{-1}(Z \times_X Z') \cong f^{-1}(Z) \times_Y f^{-1}(Z')$$

as well as étale coverings :

$$U \to V \text{ is a covering } \Rightarrow f^{-1}(U) \to f^{-1}(V) \text{ is a covering .}$$

The morphism of sites f^{-1} induces an adjoint pair of functors on the topoi. There is

$$X_{\text{ét}} \ni \mathscr{G} \mapsto f_*\mathscr{G} \in Y_{\text{ét}}, \quad (f_*\mathscr{G})(V) = \mathscr{G}\left(f^{-1}(V)\right)$$

and its left adjoint

$$Y_{\text{ét}} \ni \mathcal{G} \mapsto f^*\mathcal{G} \in X_{\text{ét}}, \quad f^*\mathcal{G} = \left(U \mapsto \varinjlim_{\substack{(u,V) \\ V \in \text{Ét}(Y) \\ u \in \text{Hom}_{\text{Ét}(X)}(U, f^{-1}(V))}} \mathcal{G}(V) \right)^{\#}$$

3.5. Étale sheaves of rings and modules. —

Définition 3.6. — An **étale sheaf of groups** on X is a group object in $X_{\text{ét}}$. An **étale sheaf of abelian groups** on X is an abelian group object in $X_{\text{ét}}$. An **étale sheaf of commutative ring** on X is a (unital) commutative ring object in $X_{\text{ét}}$.

For example, if Λ is a commutative ring, then the sheaf associated to the constant functor $Y \mapsto \Lambda$ is a sheaf of commutative ring, denoted Λ_X

If we fix an étale sheaf of commutative ring $\mathcal{A}$, then the pair $(X_{\text{ét}}, \mathcal{A})$ is a **ringed topos**.

Given such a ringed topos $(X_{\text{ét}}, \mathcal{A})$, we will denote $\text{Mod}(X_{\text{ét}}, \mathcal{A})$ the **category of $\mathcal{A}$-modules** in the obvious sense. This category is a Grothendieck abelian category, so that we can perform the construction of derived category $\text{D}(X_{\text{ét}}, \mathcal{A})$.

4. Étale cohomology

4.1. sheaves of rings and modules. — In what follows, we will only consider the case where Λ is a ring and $\mathcal{A} = \Lambda_X$. In this case, we denote simply $\text{Mod}(X, \Lambda) = \text{Mod}(X_{\text{ét}}, \Lambda_X)$ and $\text{D}(X, \Lambda) = \text{D}(X_{\text{ét}}, \Lambda_X)$.

Given any morphism $f : X \to Y$, we have a morphism of ringed topos : $(X_{\text{ét}}, \Lambda_X) \to (Y_{\text{ét}}, \Lambda_Y)$ in the sense that there are

$$f_* : X_{\text{ét}} \to Y_{\text{ét}}$$
$$f^* : Y_{\text{ét}} \to X_{\text{ét}}$$

and a morphism of étale sheaves of rings

$$f^*\left((\mathbf{Z}/\ell^n\mathbf{Z})_Y\right) \to (\mathbf{Z}/\ell^n\mathbf{Z})_X$$

4.2. étale cohomology on the spectrum of a field. — Let K be a field. We denote here $S = \text{Spec}\, K$.

Any object $T \in \text{Ét}(S)$ is a disjoint union of the form $\text{Spec}\, A$, where A is a finitely generated local étale K-algebra. It turns out that A must be a finite separable field extension of K. Therefore, fixing a separable closure K^{sep} of K, the subcategory of $\text{Ét}(S)$ consisting of spectra of finite separable sub-extensions $K \subseteq L \subseteq K^{\text{sep}}$ forms a basis for $\text{Ét}(S)$.

It is easy to deduce from the definition of étale sheaves that giving a sheaf $\mathscr{G}$ on $\text{Ét}(S)$ is equivalent to giving a discrete topological space E with a continuous $\text{Gal}(K^{\text{sep}}/K)$-action. The correspondence is given by :

$$\mathscr{G} \mapsto \varinjlim_{\substack{K \subseteq L \subseteq K^{\text{sep}} \\ \text{finite separable}}} \mathscr{G}(L)$$
$$E \mapsto \left(\text{Spec}\, L \mapsto E^{\text{Gal}(K^{\text{sep}}/L)}\right)$$

In particular, giving a sheaf of abelian groups on $\text{Ét}(S)$ is equivalent to giving a discrete $\text{Gal}(K^{\text{sep}}/K)$-module. Let E be a discrete $\text{Gal}(K^{\text{sep}}/K)$-module and let $\mathscr{G}$ be the corresponding sheaf on $\text{Ét}(S)$, we have

$$\mathrm{H}^*(S_{\text{ét}}, \mathscr{G}) \cong \mathrm{H}^*(\text{Gal}(K^{\text{sep}}/K), E).$$

Remarque 4.3. — The étale cohomology theory can be roughly divided into two parts of quite different natures : the geometric part and the arithmetic part.

The cohomology on the spectrum of a field is the "purely arithmetic" aspect of the étale cohomology. This is usually more subtle than the "purely geometric" aspect.

On the other hand, the geometric aspect is much more functorial and much closer to the classical singular cohomology theory.

Most of the time, the arithmetic part will be kept as a symmetry on schemes and on sheaves and we will only look at the "geometric cohomology".

4.4. étale cohomology on a projective smooth curve. — Now we look at a "purely geometric" example of the étale cohomology theory. Let $\mathbf{k}$ be an algebraically closed field and let X be a projective connected smooth curve on $\mathbf{k}$.

We consider firstly the subsheaf of invertible elements $\mathbf{G}_{\mathrm{m}} = \mathcal{O}_X^\times \subseteq \mathcal{O}_X$ of the étale sheaf of regular functions on X.

Théorème 4.5. — *We have*

$$\begin{aligned} \mathrm{H}^0(X_{ét}, \mathbf{G}_{\mathrm{m}}) &\cong \mathbf{k}^\times \\ \mathrm{H}^1(X_{ét}, \mathbf{G}_{\mathrm{m}}) &\cong \mathrm{Pic}(X) \\ \mathrm{H}^{\geq 2}(X_{ét}, \mathbf{G}_{\mathrm{m}}) &= 0 \end{aligned}$$

Démonstration. — For H^0, it results from the propreness of X. For H^1, one interpret $\mathrm{H}^1(X, \mathbf{G}_{\mathrm{m}})$ as the set of isomorphism classes of étale $\mathbf{G}_{\mathrm{m}}$-torsors on X. Using the theory of fpqc-descent, étale $\mathbf{G}_{\mathrm{m}}$-torsors are the same as line bundles on X. Hence $\mathrm{H}^1(X, \mathbf{G}_{\mathrm{m}}) \cong \mathrm{Pic}(X)$. For $\mathrm{H}^{\geq 2}$, one use the following short exact sequence of sheaves of abelian groups on $\text{Ét}(X)$:

$$0 \to \mathbf{G}_{\mathrm{m}} \to K(X)^\times \xrightarrow{\text{div}} \bigoplus_{x \in |X|} x_* \mathbf{Z} \to 0$$

The question is reduced to $\mathrm{H}^{\geq 2}((\operatorname{Spec} K(X))_{\text{ét}}, \mathbf{G}_{\mathrm{m}}) \cong \mathrm{H}^{\geq 2}(K(X), \mathbf{G}_{\mathrm{m}})$, whose vanishing relies on Tsen's theorem (see SGA 4 1/2 or J.-P. Serre, *Galois cohomology*). ☐

We can then calculate the cohomology of torsion sheaves with the Kummer sequence. Let $n \in \mathbf{N}^*$ be invertible in $\mathbf{k}$. Consider the short exact sequence of Kummer

$$0 \to \mu_n \to \mathbf{G}_{\mathrm{m}} \xrightarrow{n} \mathbf{G}_{\mathrm{m}} \to 0.$$

This yields

Théorème 4.6. — *We have*

$$\begin{aligned} \mathrm{H}^0(X_{ét}, \mu_n) &= \mu_n \\ \mathrm{H}^1(X_{ét}, \mu_n) &= \mathrm{Pic}^0(X)_n \\ \mathrm{H}^2(X_{ét}, \mu_n) &= \mathbf{Z}/n\mathbf{Z} \\ \mathrm{H}^{\geq 3}(X_{ét}, \mu_n) &= 0 \end{aligned}$$

where $\mathrm{Pic}^0(X)_n$ *is the group of n-torsion points of the jacobian variety of X. Thus it is isomorphic to* $(\mathbf{Z}/n\mathbf{Z})^{2g(X)}$.

4.7. base change theorems. — Let Λ be a ring such that $\Lambda = 0$ for some $N \in \mathbf{N}^*$ invertible on X. We denote $\mathrm{Mod}\,(X, \Lambda)$ the category of Λ_X-modules.

Théorème 4.8 **(base change theorem for proper morphisms).** — *Let $X \to Y$ be a proper morphism of schemes and let $g : Y' \to Y$ be a morphism of scheme. We form the cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{\widetilde{g}} & X \\ \Big\downarrow{\scriptstyle f'} & & \Big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

Then, for any sheaf $\mathscr{G} \in \mathrm{Mod}\,(X, \Lambda)$ the natural morphism

$$g^* \mathrm{R}^p f_* \mathscr{G} \to \mathrm{R}^p f'_* \widetilde{g}^* \mathscr{G}$$

is an isomorphism.

The prototypical case is when $g : Y' \to Y$ is a geometric point, in this case the theorem reads

$$(\mathrm{R}^p f_* \mathscr{G})_{\overline{y}} \cong \mathrm{H}^p \left(X_{\overline{y}}, \mathscr{G}\big|_{X_{\overline{y}}} \right).$$

To realise what this theorem is special about, one can compare it with the formal function theorem in the theory of coherent sheaves.

Théorème 4.9 **(local acyclicity theorem for smooth morphisms)**

Let $X \to Y$ be a smooth morphism of schemes. Then the sheaf Λ_X is universally locally f-acyclic. That is, for any $g : Y' \to Y$ be a qcqs morphism of scheme if we form the cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{\widetilde{g}} & X \\ \Big\downarrow{\scriptstyle f'} & & \Big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

then, for any sheaf $\mathscr{G} \in \mathrm{Mod}\,(Y', \Lambda)$ the natural morphism

$$f^* \mathrm{R}^p g_* \mathscr{G} = \Lambda_X \otimes f^* \mathrm{R}^p g_* \mathscr{G} \to \mathrm{R}^p \widetilde{g}_* \left(\widetilde{g}^* \Lambda_X \otimes f'^* \mathscr{G} \right) = \mathrm{R}^p \widetilde{g}_* f'^* \mathscr{G}$$

is an isomorphism.

Exemple 4.10. — Let A be a strict henselian local ring, $S = \operatorname{Spec} A$ the spectrum, $s \in S$ the closed point and $\overline{\eta} \in S$ a geometric point. Let $f : X \to S$ be a smooth proper morphism, then we have a diagram of cartesian squares

$$\begin{array}{ccccc} X_s & \longrightarrow & X & \xleftarrow[\widetilde{g}]{} & X_{\overline{\eta}} \\ \Big\downarrow{\scriptstyle f_s} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f_{\overline{\eta}}} \\ \{s\} & \longrightarrow & S & \xleftarrow[g]{} & \{\overline{\eta}\} \end{array}$$

We know that

$$(\mathrm{R}^p f_* \Lambda_X)_s \cong \mathrm{H}^p (X_s, \Lambda_{X_s}).$$

by base change theorem for proper morphisms, and that

$$\Lambda_X \cong f_{\overline{\eta}}^* g_* \Lambda_{\overline{\eta}} \cong \widetilde{g}_* f_{\overline{\eta}}^* \Lambda_{\overline{\eta}} \cong \widetilde{g}_* \Lambda_{X_{\overline{\eta}}}$$

Thus

$$(\mathrm{R}^p f_* \Lambda_X)_s \cong \left(\mathrm{R}^p f_* \widetilde{g}_* \Lambda_{X_{\overline{\eta}}} \right)_s$$

Since $\overline{\eta}$ is a geometric point, the functor g_* is exact, so

$$\left(\mathrm{R}^p f_* \widetilde{g}_* \Lambda_{X_{\overline{\eta}}} \right)_s \cong \left(g_* \mathrm{R}^p f_{\eta *} \Lambda_{X_{\overline{\eta}}} \right)_s \cong \left(g_* \mathrm{R}^p f_{\eta *} \Lambda_{X_{\overline{\eta}}} \right)(S) \cong \mathrm{H}^p \left(X_{\overline{\eta}}, \Lambda_{X_{\overline{\eta}}} \right).$$

We found that $\mathrm{H}^p (X_s, \Lambda_{X_s}) \cong \mathrm{H}^p \left(X_{\overline{\eta}}, \Lambda_{X_{\overline{\eta}}} \right)$. This is an étale analogue of the Ehresmann theorem.

Exemple 4.11. — In the setting of the previous example, if now f is proper but not necessarily smooth and if $\mathscr{K}^\cdot \in \mathrm{D}(X, \Lambda)$, then we still have a **cospecialisation** morphism

$$\mathrm{sp}^* : \mathrm{H}^p\left(X_s, \mathscr{K}^\cdot|_{X_s}\right) \to \mathrm{H}^p\left(X_{\overline{\eta}}, \mathscr{K}^\cdot|_{X_{\overline{\eta}}}\right).$$

Let η be the image of $\overline{\eta}$ in S. Then $X_{\overline{\eta}} = X_\eta \times_\eta \overline{\eta}$. It follows from the functoriality that the cospecialisation factorises as

$$\mathrm{H}^p\left(X_s, \mathscr{K}^\cdot|_{X_s}\right) \to \mathrm{H}^p\left(X_\eta, \mathscr{K}^\cdot|_{X_\eta}\right) \to \mathrm{H}^p\left(X_{\overline{\eta}}, \mathscr{K}^\cdot|_{X_{\overline{\eta}}}\right).$$

There is an action of the galois group $\mathrm{Gal}(\overline{\eta}, \eta)$ on $X_{\overline{\eta}}$, on $\mathscr{K}^\cdot|_{X_{\overline{\eta}}}$ and thus on $\mathrm{H}^p\left(X_{\overline{\eta}}, \mathscr{K}^\cdot|_{X_{\overline{\eta}}}\right)$. Again, by functoriality, the image of $\mathrm{H}^p\left(X_\eta, \mathscr{K}^\cdot|_{X_\eta}\right) \to \mathrm{H}^p\left(X_{\overline{\eta}}, \mathscr{K}^\cdot|_{X_{\overline{\eta}}}\right)$ falls into the invariants $\mathrm{H}^p\left(X_{\overline{\eta}}, \mathscr{K}^\cdot|_{X_{\overline{\eta}}}\right)^{\mathrm{Gal}(\overline{\eta}/\eta)}$. Thus, we have

$$\mathrm{sp}^* : \mathrm{H}^p\left(X_s, \mathscr{K}^\cdot|_{X_s}\right) \to \mathrm{H}^p\left(X_{\overline{\eta}}, \mathscr{K}^\cdot|_{X_{\overline{\eta}}}\right)^{\mathrm{Gal}(\overline{\eta}/\eta)}.$$

One of the applications of the theory of weights gives a sufficient condition for this morphism to be surjective. This is the **local invariant cycle theorem**.

4.12. constructibility. — Suppose now that Λ is a self-injective noetherian ring.

Définition 4.13. — A sheaf of Λ-modules $\mathscr{G} \in \mathrm{Mod}(X, \Lambda)$ is called **constructible** if it is a noetherian object in that category. It is called of **finite tor-dimension** if there exists an integer $n \in \mathbf{N}$ such that $\mathrm{Tor}_{\geq n}(\mathscr{G}, M_X) = 0$ for any Λ-module M.

Let $\mathrm{D}^b_{\mathrm{ctf}}(X, \Lambda) \subseteq \mathrm{D}(X, \Lambda)$ be the full subcategory consisting of complexes $\mathscr{K}^\cdot$ of bounded cohomology, such that each cohomology sheaf $\mathscr{H}^k(\mathscr{K}^\cdot)$ is constructible and of finite tor-dimension.

Théorème 4.14. — *Let X and Y be schemes of finite type over a field $\mathbf{k}$ and let $f : X \to Y$ be a morphism. Then the functors*

$$\mathrm{R}f_* : \mathrm{D}(X, \Lambda) \to \mathrm{D}(Y, \Lambda)$$

send $\mathrm{D}^b_{\mathrm{ctf}}(X, \Lambda)$ to $\mathrm{D}^b_{\mathrm{ctf}}(Y, \Lambda)$. Similarly,

$$\mathrm{R}f^* : \mathrm{D}(Y, \Lambda) \to \mathrm{D}(X, \Lambda)$$

send $\mathrm{D}^b_{\mathrm{ctf}}(Y, \Lambda)$ to $\mathrm{D}^b_{\mathrm{ctf}}(X, \Lambda)$.

Moreover,

Théorème 4.15. — *Let X be a scheme of finite type over a field $\mathbf{k}$ then there is a biduality functor*

$$\mathbf{D}_X : \mathrm{D}^b_{\mathrm{ctf}}(X, \Lambda)^{\mathrm{op}} \to \mathrm{D}^b_{\mathrm{ctf}}(X, \Lambda)$$

sending $\mathscr{G}$ to $\mathbf{D}_X\mathscr{G} = \mathrm{R}\mathscr{H}\mathrm{om}\left(\mathscr{G}, \omega_{X/\mathbf{k}}\right)$, together with a natural isomorphism $\mathrm{id} \cong \mathbf{D}_X \circ \mathbf{D}_X$.

4.16. derived category of constructible ℓ-adic sheaves. — Let X be a scheme of finite type over a field $\mathbf{k}$ and let ℓ be a prime number which is invertible in $\mathbf{k}$. We define

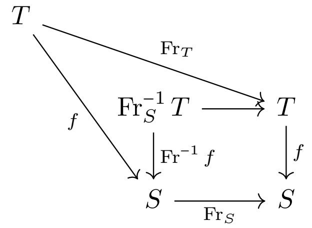
$$\mathrm{D}^b_c\left(X, \overline{\mathbf{Q}}_\ell\right) = \varinjlim_{\substack{F/\mathbf{Q}_\ell \\ \text{finite extension}}} \left(\varprojlim_{k \geq 1} \mathrm{D}^b_{\mathrm{ctf}}\left(X, \mathcal{O}_F/\ell^k\right)\right) \otimes_{\mathcal{O}_F} F.$$

where $\mathcal{O}_F$ is the integral closure of $\mathbf{Z}_\ell$ in F. Then $\mathrm{D}^b_c\left(X, \overline{\mathbf{Q}}_\ell\right)$ is a triangulated category (this uses the constructibility), and we have the six operations $f_*, f^*, f_!, f^!, \mathscr{H}\mathrm{om}, \otimes$ on these categories.

5. Theory of weights for ℓ-adic sheaves

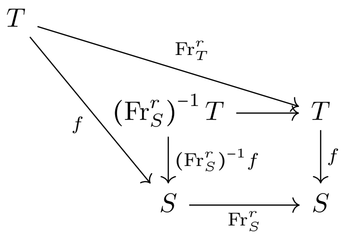
Let X_0 be an algebraic variety over $\mathbf{F}_q$ and let $\mathbf{k} = \overline{\mathbf{F}}_q$ be an algebraic closure of $\mathbf{F}_q$. We denote $X = X_0 \otimes_{\mathbf{F}_q} \mathbf{k}$.

5.1. geometric Frobenius. — For any scheme S on which $p = 0$, there is an **absolute Frobenius morphism** $\mathrm{Fr}_S : S \to S$. The morphism Fr_S induces the identity map on the underlying topological space but raising any regular function in $\mathcal{O}_S(U)$ to its p-th power $a \mapsto a^p$. If $f : T \to S$ is a scheme over S, then we can form the following diagram in which the bottom-right square is cartesian :



The universal property of fiber product then gives a morphism $\mathrm{Fr}_{T/S} : T \to \mathrm{Fr}_S^{-1} T$ over S. This is called the **relative Frobenius morphism** of T/S.

Iterating r times, we get



as well as $\mathrm{Fr}_{T/S}^r : T \to \mathrm{Fr}_S^{-r} T$.

Assume now that $S = \operatorname{Spec} \mathbf{F}_q$ and $T = X_0$ where $q = p^r$. Then $\mathrm{Fr}_S^r = \mathrm{id}_S$, so we get a morphism over $\mathbf{F}_q$

$$\mathrm{Fr}^r_{X_0/\mathbf{F}_q} : X_0 \to X_0.$$

Définition 5.2. — The above morphism

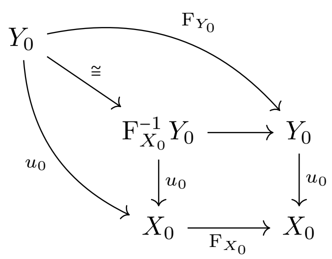
$$\mathrm{F}_{X_0} = \mathrm{Fr}^r_{X_0/\mathbf{F}_q} : X_0 \to X_0.$$

and the its base change to $\mathbf{k}$

$$\mathrm{F}_X = \mathrm{F}_{X_0} \otimes_{\mathbf{F}_q} \mathbf{k}$$

are called the **geometric endomorphism** of X.

5.3. Weil sheaves. — We keep the previous notations. Let $Y_0 \in \text{Ét}(X_0)$ be an object with the étale structural morphism $u_0 : Y_0 \to X_0$. In this case, the following square is cartesian



For any sheaf $\mathscr{G}_0$ on $\text{Ét}(X_0)$, we have a canoncial morphism $\mathrm{F}_* : \mathrm{F}_0^* \mathscr{G}_0 \to \mathscr{G}_0$ defined in the following way : for any $Y_0 \in \text{Ét}(X_0)$ with $u_0 : Y_0 \to X_0$, we take the composite

$$\mathscr{G}(Y_0) \cong \mathscr{G}\left(\mathrm{F}_{X_0}^{-1} Y_0\right) \cong \left(\mathrm{F}_{X_0,*} \mathscr{G}\right)(Y_0)$$

This yields a natural isomorphism of sheaves $\mathscr{G}_0 \cong \mathrm{F}_{X_0,*} \mathscr{G}_0$. Thus by adjunction, we have a natural isomorphism

$$\mathrm{F}_{\mathscr{G}_0} : \mathrm{F}_{X_0}^* \mathscr{G}_0 \cong \mathscr{G}_0.$$

By by change to $\mathbf{k}$, we obtain a sheaf $\mathscr{G}$ on $\text{Ét}(X)$ and

$$\mathrm{F}_{\mathscr{G}} : \mathrm{F}_X^* \mathscr{G} \cong \mathscr{G}.$$

This construction carries out on the category of étale $\mathbf{Z}/\ell^k$-sheaves $\mathrm{Mod}\left(X_0, \mathbf{Z}/\ell^k\mathbf{Z}\right)$ and subsequently on

$$\mathrm{Mod}\left(X_0, \overline{\mathbf{Q}}_\ell\right).$$

Thus to any ℓ-adic étale sheaf $\mathscr{G}_0$ on X_0, we can attach a datum $(\mathscr{G}, \mathrm{F}_\mathscr{G})$ where $\mathscr{G}$ is an ℓ-adic sheaf on X and $\mathrm{F}_\mathscr{G} : \mathrm{F}_X^* \mathscr{G} \to \mathscr{G}$ is an isomorphism. It turns out that this datum suffices to determine $\mathscr{G}_0$.

We define

Définition 5.4. — A **Weil sheaf** $\mathscr{G}_0$ on X_0 is an ℓ-adic étale sheaf $\mathscr{G} \in \mathrm{Mod}\left(X, \overline{\mathbf{Q}}_\ell\right)$ together with an isomorphism

$$\mathrm{F}_\mathscr{G} : \mathrm{F}_X^* \mathscr{G} \xrightarrow{\cong} \mathscr{G}$$

We get thus a fully faithful functor

$$\mathrm{Mod}\left(X_0, \overline{\mathbf{Q}}_\ell\right) \to \mathrm{Weil}\left(X_0, \overline{\mathbf{Q}}_\ell\right).$$

We will be focusing on Weil sheaves.

Exemple 5.5. — Let K be a finite extension of $\mathbf{F}_q$ and we denote $d = [K : \mathbf{F}_q]$. Let K^{sep} be a separable closure of K so that we have a geometric point $\overline{x} \in X_0(K^{\mathrm{sep}})$. The relative Frobenius $\mathrm{F}_{\mathrm{Spec}\, K^{\mathrm{sep}}/\mathrm{Spec}\, K} : \mathrm{Spec}\, K^{\mathrm{sep}} \to \mathrm{Spec}\, K^{\mathrm{sep}}$ acts on the field K^{sep} as the the canonical generator

$$1 \in \widehat{\mathbf{Z}} \cong \mathrm{Gal}\left(K^{\mathrm{sep}}/K\right).$$

We should think of it rather as in the Weil group $\mathrm{W}\left(k(x)^{\mathrm{sep}}/k(x)\right) = \mathbf{Z} \subseteq \widehat{\mathbf{Z}} = \mathrm{Gal}\left(k(x)^{\mathrm{sep}}/k(x)\right)$. Given a Weil sheaf $\mathscr{G}_0$ on $X_0 = \mathrm{Spec}\, K$, we have then an endomorphism on the stalk

$$\mathscr{G}_{0,\overline{x}} \cong \mathscr{G}_{0,\mathrm{F}_{X_0}^{-1}\overline{x}} \xrightarrow{\mathrm{F}_{\mathscr{G}_0}} \mathscr{G}_{0,\overline{x}},$$

denoted $\mathrm{F}_x : \mathscr{G}_{0,\overline{x}} \to \mathscr{G}_{0,\overline{x}}$.

In general, if X_0 is a scheme over $\mathbf{F}_q$ and if $x \in |X_0|$ is the image of a geometric point $\overline{x} : \mathrm{Spec}\, \overline{k}(x)^{\mathrm{sep}} \to X_0$, we have a canonical action of the Weil group $\mathrm{W}\left(k(x)^{\mathrm{sep}}/k(x)\right) \cong \mathbf{Z}$ on the stalk $\mathscr{G}_{0,\overline{x}}$ of any Weil sheaf $\mathscr{G}_0$ on X_0, given by $F_x : \mathscr{G}_0 \to \mathscr{G}_0$.

5.6. weights. — Let $\tau : \overline{\mathbf{Q}}_\ell \hookrightarrow \mathbf{C}$ be an embedding. Let $\mathscr{G}_0 = (\mathscr{G}, \mathrm{F}_\mathscr{G})$ be an ℓ-adic Weil sheaf.

Définition 5.7. — In the above notation,

(i). We say that $\mathscr{G}_0$ is **τ-pure of weight** $\beta \in \mathbf{R}$ if for each $x \in |X_0|$ and for each eigenvalue $\lambda \in \overline{\mathbf{Q}}_\ell$ of the endomorphism

$$\mathrm{F}_x : \mathscr{G}_{0,\overline{x}} \to \mathscr{G}_{0,\overline{x}}$$

we have

$$|\tau(\lambda)| = (\#k(x))^{\beta/2}$$

(ii). We say that $\mathscr{G}_0$ is **pure of weight** $\beta \in \mathbf{R}$ if it is τ-pure of weight β for any embedding $\tau : \overline{\mathbf{Q}}_\ell \to \mathbf{C}$.

(iii). We say that $\mathscr{G}_0$ is **mixed** if it admits a filtration $0 = F_0\mathscr{G}_0 \subseteq \cdots \subseteq F_m\mathscr{G}_0 = \mathscr{G}_0$ such that each successive quotient $F_i\mathscr{G}_0/F_{i-1}\mathscr{G}_0$ is pure.

Exemple 5.8. — Let K be a finite extension of $\mathbf{F}_q$ with $[K : \mathbf{F}_q] = d$ and let $T_\ell = \varprojlim_{k \geq 0} \mu_{\ell^k} \in \mathrm{Mod}\left(\mathrm{Spec}\, \mathbf{F}_q, \mathbf{Z}_\ell\right)$ be the ℓ-adic Tate module. Denoting $\overline{x} = \mathrm{Spec}\, \overline{K}$, then we have

$$\mathrm{F}_{T_\ell} : T_{\ell,\overline{K}} \xrightarrow{\left(\mathrm{Fr}^*_{\mathrm{Spec}\,\overline{K}/\mathrm{Spec}\,K}\right)^{-1} : \zeta \mapsto \zeta^{1/q^d}} T_{\ell,\overline{K}}$$

Denote $\overline{\mathbf{Q}}_\ell(1) = \overline{\mathbf{Q}}_\ell \otimes_{\mathbf{Z}_\ell} T_\ell \in\in \mathrm{Mod}\left(\mathrm{Spec}\, \mathbf{F}_q, \overline{\mathbf{Q}}_\ell\right)$, we see that the only eigenvalue of $\mathrm{F}_{\overline{x}} : \overline{\mathbf{Q}}_\ell(1) \to \overline{\mathbf{Q}}_\ell(1)$ is $q^{-d} = (\#K)^{-1}$. Thus $\overline{\mathbf{Q}}_\ell(1)$ is pure of weight -2. This generalises to any scheme X_0 over $\mathbf{F}_q$.

Exemple 5.9. — Let A_0 be an abelian variety over $\mathbf{F}_q$, $A = A_0 \otimes_{\mathbf{F}_q} \mathbf{k}$ and let $T_\ell(A_0) = \varprojlim_{k\geq 0} A\left[\ell^k\right]$ be the ℓ-adic Tate module of A_0, viewed as étale sheaf on $\operatorname{Spec}\mathbf{F}_q$. We have

$$\mathrm{F}_{T_\ell} : T_\ell(A_0) \xrightarrow{x\mapsto x^{1/q}} T_\ell(A_0).$$

We put $V_\ell(A_0) = T_\ell(A_0)\otimes_{\mathbf{Z}_\ell} \in \overline{\mathbf{Q}}_\ell \operatorname{Mod}\left(\operatorname{Spec}\mathbf{F}_q, \overline{\mathbf{Q}}_\ell\right)$. There is a skew-symmetric non-degenerate pairing

$$V_\ell(A_0) \times V_\ell\left(\widehat{A}_0\right) \to \overline{\mathbf{Q}}_\ell(1).$$

where $\widehat{A}_0$ is the dual abelian variety. Fixing a polarisation of A_0, which induces a $\mathbf{Q}$-isogeny $A_0 \to \widehat{A}_0$. We get a perfect pairing

$$V_\ell(A_0) \times V_\ell(A_0) \to \overline{\mathbf{Q}}_\ell(1).$$

This action is Frobenius-equivariant. Using this pairing and Rosati involution on $\operatorname{End}^0 A_0$, André Weil showed that $V_\ell(A_0)$, as étale ℓ-adic sheaf on $\operatorname{Spec}\mathbf{F}_q$, is pure of weight -1.

The reason that we have weight -1 instead of 1 as we know from number theory is that our Frobenius F_{T_ℓ} acts as the inverse to the usual "arithmetic Frobenius" on $A_0(\mathbf{k})$ which raises each coordinate to the q-th power.

5.10. Deligne's theorem Weil II. — The fundamental theorem of the theory of weights is the following

Théorème 5.11 **(Deligne)**. — *Given a morphism $f_0 : X_0 \to Y_0$ of algebraic variety over $\mathbf{F}_q$ and let $\mathscr{G}_0$ be a mixed ℓ-adic Weil sheaf of weight $\leq \beta$. Then for all $i \geq 0$, the sheaf $R^i f_{0,!}\mathscr{G}_0$ is mixed of weight $\leq \beta + i$.*

Using Verdier biduality, we obtain

Corollaire 5.12 **(Weil conjectures – Riemann hypothesis)**. — *Suppose that $f_0 : X_0 \to Y_0$ is smooth and proper and that $\mathscr{G}_0$ is pure of weight β. Then $R^i f_{0,*}\mathscr{G}_0$ is pure of weight $\beta + i$.*

Remarque 5.13. — The failure for a similar result for $R^i f_*$ comes from the singularity of f_0 : the Verdier duality functor $\mathbf{D}_{X_0} : \mathrm{D}^b_c\left(X_0, \overline{\mathbf{Q}}_\ell\right)$ does not commutes with stalks : $\mathscr{G}_0 \mapsto \mathscr{G}_{0,\overline{x}}$ even up to shift. In order to have a good theory of weights for $f_{0,*}$, we should take the "strict stalk" functor $\overline{x}^! : \mathrm{D}^b_c\left(X_0, \overline{\mathbf{Q}}_\ell\right) \to \mathrm{D}^b_c\left(\overline{\mathbf{Q}}_\ell\right)$ into account. This leads to the introduction of mixed perverse sheaves.

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4 avril 2018

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