NOTES OF 9 MAI 2018 : CLASSIFICATION OF EQUIVARIANT MONODROMIC $\mathcal{D}$-MODULES

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1. Langlands classification of equivariant $\mathcal{D}$-modules

1.1. Equivariant H-torsors. — Let G be another algebraic group with $\pi_0(G)$ acting on H via $\kappa : \pi_0(G) \to \operatorname{Aut} H$. Let $\widetilde{X}$ be a $G \ltimes H$-variety such that $\pi : \widetilde{X} \to X$ is an H-torsor. This action induces a G-action on X. In other words, $[G \backslash \widetilde{X}]$ is an H-torsor on $[G \backslash X]$.

We suppose that X has only a finite number of G-orbits.

We shall study the category $\mathrm{Mod}_{\mathrm{fin}}\left(\widetilde{\mathcal{D}}_X, G\right)$.

Recall : centre $\mathrm{U}(\mathfrak{h}) \subseteq \widetilde{\mathcal{D}}_X$. The subcategory $\mathrm{Mod}_{\mathrm{fin}}\left(\widetilde{\mathcal{D}}_X, G\right)$ consists of those modules on which $\mathrm{U}(\mathfrak{h})$ acts locally finitely (i.e. every local section is annihilated by an ideal of $\mathrm{U}(\mathfrak{h})$ of finite codimension).

1.2. Action on π-fibres. — Let $G^\kappa = \ker \kappa \subseteq G$ be the subgroup which centralises H. For each $x \in X$, the stabiliser $G^\kappa_x = \mathrm{Stab}_{G^\kappa}(x)$ acts on the fibre $\widetilde{X}_x = \pi^{-1}(x) \subseteq \widetilde{X}$ which commutes with the H-action.

Since $\widetilde{X}_x$ is a principal homogeneous H-space, there is a homomorphism $\varphi_x : G^\kappa_x \to H$ such that the G^κ_x-action on $\pi^{-1}(x)$ agrees with the action via φ_x and H. Clearly, $\ker \varphi_x = \mathrm{Stab}_G(\widetilde{x}) = G_{\widetilde{x}}$ for any $\widetilde{x} \in \widetilde{X}_x$.

We define $G_{(x)} = G_x / G^0_{\widetilde{x}}$ and $\mathfrak{g}_{(x)} = \operatorname{Lie} G_{(x)}$. The homomorphism φ_x induces $\mathfrak{g}_{(x)} \hookrightarrow \mathfrak{h}$.

1.3. Case : homogeneous G-space. — Suppose that $X = Gx$ for $x \in X$. Then $\widetilde{X}_x \to x$ is an H-monodromic G_x-variety.

Lemme 1.4. — *There is an equivalence*

$$\mathrm{Mod}\left(\widetilde{D}_X, G\right) \cong \mathrm{Mod}\left(\mathrm{U}(\mathfrak{h}), G_{(x)}\right)$$

Démonstration. — Using the isomorphisms $\left[G\backslash\widetilde{X}\right] \cong \left[G_x\backslash\widetilde{X}_x\right]$ and $[G\backslash X] \cong [G_x\backslash\{x\}]$

$$\begin{aligned}\mathrm{Mod}\left(\widetilde{D}_X, G\right) &\cong \mathrm{Mod}\left(\widetilde{D}_{[G\backslash X]}\right) \cong \mathrm{Mod}\left(\widetilde{D}_{[G_x\backslash\{x\}]}\right)\\ &\cong \mathrm{Mod}\left(\mathrm{U}\left(\mathfrak{h}\right), G_x\right)\end{aligned}$$

Since for any $M \in \mathrm{Mod}(\mathrm{U}(\mathfrak{h}), G_x)$, the $\mathfrak{g}_x$-action on M from $d\varphi_x : \mathfrak{g}_x \to \mathfrak{h}$ agrees the one coming from the G_x action, it is clear that the G_x action factorises through $G_x/\left(\ker\varphi_x\right)^0 = G_{(x)}$. Therefore

$$\mathrm{Mod}\left(\mathrm{U}\left(\mathfrak{h}\right), G_x\right) \cong \mathrm{Mod}\left(\mathrm{U}\left(\mathfrak{h}\right), G_{(x)}\right).$$

□

1.5. General case. — Suppose in general that X has a finite number of G-orbits. Let $X = \sqcup_{i\in I} Q_i$ be the decomposition into orbits.

For $i_1, i_2 \in I$, we write $i_1 \le i_2$ if $Q_{i_1} \subseteq \overline{Q}_{i_2}$. A subset $J \subseteq I$ is called closed if $i_1 \le i_2$ and $i_2 \in J$ implies i_1.

For any closed $J \subseteq I$, we denote $X_J = \bigcup_{i\in J} Q_i$ and $\mathscr{C}_J = \mathrm{Mod}\left(\widetilde{\mathcal{D}}_{X_J}, G\right)$ or $\mathrm{Mod}_{\mathrm{fin}}\left(\widetilde{\mathcal{D}}_{X_J}, G\right)$ or $\mathrm{Mod}\left(\widetilde{\mathcal{D}}_{X_J}, G\right)_\lambda$ or $\mathrm{Mod}\left(\widetilde{\mathcal{D}}_{X_J}, G\right)_{\widetilde{\lambda}}$.

Here are some properties of the family of categories $\{\mathscr{C}_J\}_{J\subseteq I}$:

(i). $\mathscr{C}_J \subseteq \mathscr{C}_I$ is a Serre subcategory (stable under extensions, sub-objects and quotients).

(ii). For two closed subsets J_1 and J_2, $\mathscr{C}_{J_1\cap J_2} = \mathscr{C}_{J_1} \cap \mathscr{C}_{J_2}$ and $\mathscr{C}_{J_1\cup J_2}$ is the Serre subcategory generated by $\mathscr{C}_{J_1}$ and $\mathscr{C}_{J_2}$.

(iii). For two closed subsets J_1 and J_2, the square

$$\begin{array}{ccc}\mathscr{C}_{J_1\cap J_2} & \hookrightarrow & \mathscr{C}_{J_1}\\ \downarrow & & \downarrow\\ \mathscr{C}_{J_2} & \hookrightarrow & \mathscr{C}_{J_1\cup J_2}\end{array}$$

induces an equivalence

$$\left(\mathscr{C}_{J_1}/\mathscr{C}_{J_1\cap J_2}\right) \times \left(\mathscr{C}_{J_2}/\mathscr{C}_{J_1\cap J_2}\right) \cong \left(\mathscr{C}_{J_1\cup J_2}/\mathscr{C}_{J_1\cap J_2}\right)$$

(iv). For any $J_1 \subseteq J_2$, the projection $\mathscr{C}_{J_2} \to \mathscr{C}_{J_2}/\mathscr{C}_{J_1}$ admits a left adjoint $j_{J_2\smallsetminus J_1,!} : \mathscr{C}_{J_2}/\mathscr{C}_{J_1} \to \mathscr{C}_{J_2}$ and a right adjoint $j_{J_2\smallsetminus J_1,*} : \mathscr{C}_{J_2}/\mathscr{C}_{J_1} \to \mathscr{C}_{J_2}$.

Proposition 1.6. — *There is a homomorphism* $j_{J_2\smallsetminus J_1,!} \to j_{J_2\smallsetminus J_1,*}$ *and its image* $j_{J_2\smallsetminus J_1,!*} = \mathrm{im}\left(j_{J_2\smallsetminus J_1,!} \to j_{J_2\smallsetminus J_1,*}\right)$ *sends irreducibles to irreducibles. Therefore, it is exact.*

Corollaire 1.7. — *(i). For each G-orbit* $X_i \subseteq X$ *and* $x \in X_i$*, there is an equivalence* $\mathrm{Mod}\left(\mathcal{D}_{\widetilde{X}_i}, G\right) \cong \mathrm{Mod}\left(\mathrm{U}\left(\mathfrak{h}\right), G_{(x)}\right)$

(ii). The categories $\mathrm{Mod}_{\mathrm{fin,h}}\left(\widetilde{\mathcal{D}}_X, G\right)$ *and* $\mathrm{Mod}_{\mathrm{fin,rh}}\left(\mathcal{D}_{\widetilde{X}}, G\right)$ *satisfies the Jordan–Hölder property. The irreducible objects of it are parametried by a pair of* $i \in I$ *and irreducible object* $V \in \mathrm{Mod}\left(\mathrm{U}\left(\mathfrak{h}\right), G_{(x)}\right)$.

(iii). Every object in $\mathrm{Mod}_{\mathrm{fin,h}}\left(\widetilde{\mathcal{D}}_X, G\right)$ *is tame (has regular singularities). If* $\lambda^{\cdot} \in \pi_0(G)\backslash\mathfrak{h}^*_{\mathbf{Q}}/\mathfrak{h}^*_{\mathbf{Z}}$, *then any object* $\mathrm{Mod}\left(\mathcal{D}_{\widetilde{X}}, G\right)_{\lambda^{\cdot}}$ *is of geometric origin.*

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