LECTURES ON GROMOV–WITTEN THEORY AND CREPANT TRANSFORMATION CONJECTURE

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ABSTRACT. These are the pre-notes for the Grenoble Summer School lectures June-July 2011. They aim to provide the students some background in preparation for the conference. Nominally, only the basic knowledge on moduli of curves covered in the first week is assumed, although I tacitly assume the students either have heard one thing or two about the subjects, or are formidable learners. It is well-nigh impossible for a mere human to learn GWT in a week. In fact, I only know of two persons who have done it.

Please help find the errors, typographical or mathematical. Without a moment's hesitation, I bet there are plenty.

CONTENTS

1. DEFINING GWI

The ground field is $\mathbb{C}$. All cohomological degrees are Chow or "complex" degrees, and dimensions are complex dimensions.

1.1. **Moduli of stable maps.**

The main reference is [4].[1]

An n-pointed, genus g, *prestable curve* $(C, x_1, x_2, \ldots, x_n)$ is a projective, connected, reduced, nodal curve of arithmetic genus g with n distinct, nonsingular, ordered marked points.

Let S be an algebraic scheme. (A family of) n-pointed, genus g, prestable curve over S is a flat projective morphism $\pi : \mathcal{C} \to S$ with n sections

[1]The references of these pre-notes are mostly survey articles. For those interested in the original papers, please ask the experts. We have many in the School!

$x_1, x_2, \ldots, x_n$, such that every geometric fiber is an n-pointed, genus g, prestable curve defined above.

Let X be an algebraic scheme. A *prestable map* over S from n-pointed, genus g curves to X is the following diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & X \\ {\scriptstyle\pi}\big\downarrow & & \\ S & & \end{array}$$

such that π is described above and f is a morphism. Two maps $f_i : \mathcal{C}_i \to X$ over S are isomorphic if there is an isomorphism $g : \mathcal{C}_1 \to \mathcal{C}_2$ *over* S such that $f_1 \circ g \cong f_2$.

A prestable map *over* $\mathbb{C}$ is called *stable* if it has no infinitesimal automorphism. A prestable map *over* S is called *stable* if the map on each geometric fiber of π is stable.

Exercise 1.1. Prove that the stability condition is equivalent to the following: For every irreducible component $C_i \subset C$,

(1) if $C_i \cong \mathbb{P}^1$ and C_i maps to a point in X, then C_i contains at least 3 special (nodal and marked) points;
(2) if the arithmetic genus of C_i is 1 and C_i maps to a point, then C_i contains at least 1 special point.

To form a moduli stack of finite type, one needs to fix another topological invariant $\beta := f_*([C]) \in NE(X)$ in addition to g and n, where $NE(X)$ stands for Mori cone of the numerical (or homological) classes of effective 1 cycles. Let $\overline{M}_{g,n}(X, \beta)$ be the moduli stack of the functor defined above.

Theorem 1.2 (Kontsevich (i), Pandharipande (ii))**.** *The moduli* $\overline{M}_{g,n}(X, \beta)$

(i) *is a proper separated Deligne–Mumford stack of finite type (over* $\mathbb{C}$*), and*
(ii) *has a projective coarse moduli scheme.*

1.2. **Natural morphisms.** As in moduli of curves, there are the *forgetful morphisms*

$$\mathrm{ft}_i : \overline{M}_{g,n+1}(X, \beta) \to \overline{M}_{g,n}(X, \beta),$$

forgetting the i-th marked point and stabilize. As you must have learned in the First Week of the School, the above "set-theoretic" description can be made functorial. In fact,

Exercise 1.3. $\mathrm{ft}_{n+1} : \overline{M}_{g,n+1}(X, \beta) \to \overline{M}_{g,n}(X, \beta)$ is isomorphic to the *universal curve*. (This is similar to the case of curves.)

The *evaluation morphisms*

$$\mathrm{ev}_i : \overline{M}_{g,n}(X, \beta) \to X,$$

are the morphisms which send $[f : (C, x_1, \ldots, x_n) \to X]$ to $f(x_i) \in X$.

The *stabilization morphism*

$$\mathrm{st} : \overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,n}$$

exists when $\overline{M}_{g,n}$ does. It assigns an (equivalence class of) stable curve $[(\bar{C}, \bar{x_1}, \ldots, \bar{x_n})]$ to (that of) a stable map $[f : (C, x_1, \ldots, x_n) \to X]$. Some stabilization might be necessary to ensure the stability of the pointed curve $[(\bar{C}, x_1, \ldots, x_n)]$.

Exercise 1.4. Formulate the stabilization for the forgetful morphisms which forgets one marked point. (Probably done in the first week already.)
Hint: In terms of families over S:

$$\bar{\mathcal{C}} \hookrightarrow \mathbf{Proj}_S \left(\oplus_{k=0}^{\infty} \pi_* \left((\omega_{\mathcal{C}/S}(\sum_i x_i))^{\otimes k} \right) \right),$$

where $\omega_{\mathcal{C}/S}$ is the dualizing line bundle.

As in the case of moduli of curves, there are also *gluing morphisms*:

$$\sum_{\beta'+\beta''=\beta} \sum_{n'+n''=n} \overline{M}_{g_1,n'+1}(X,\beta') \times_X \overline{M}_{g-g_1,n''+1}(X,\beta'') \to \overline{M}_{g,n}(X,\beta), \tag{1.1}$$

and

$$\begin{array}{ccccc} \overline{M}_{g-1,n+2}(X,\beta) & \longleftarrow & D & \longrightarrow & \overline{M}_{g,n}(X,\beta). \\ \downarrow & & \downarrow & & \\ X \times X & \xleftarrow[\Delta]{} & X & & \end{array} \tag{1.2}$$

Remark 1.5. The images of the gluing morphisms are in the "boundary" of the the moduli. However, unlike the curve theory, the moduli are not of pure dimension in general, and it doesn't really make sense to talk about the "divisors". On the other hand, the virtual classes, which we will talk about soon albeit in a superficial way, are compatible with the gluing morphisms. Thus the gluing defines virtual divisors.

1.3. **Gromov–Witten invariants and the axioms.** Given a projective smooth variety X, Gromov–Witten invariants (GWIs) for X are numerical invariants constructed via the auxiliary moduli spaces/stacks $\overline{M}_{g,n}(X,\beta)$. They are called invariants because they are (real) *symplectic-deformation invariants* of X. We will say say nothing about symplectic perspective but to point out that it does mean that GWIs are deformation invariant. Even though the spaces are proper, of finite type, they are usually singular and badly behaved. In fact, they can be as badly behaved as any prescribed singularities. (This is R. Vakil's "Murphy's Law".)

However, these GWIs will behave mostly like they are defined via smooth auxiliary spaces, thanks to the existence of and the functorial properties enjoyed by the *virtual fundamental classes* $[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}$. A good, concise exposition of the construction of the virtual classes can be found in the first

few pages of [7] and will not be repeated here.[2] Instead, we will only state some functorial properties (or *axioms*) these invariants, or equivalently the virtual classes, have to satisfy.

One of the most important properties of the virtual class is the *virtual dimension* (or expected dimension, or Riemann–Roch dimension...)

$$\mathrm{vdim}(\overline{M}_{g,n}(X,\beta)) := -K_X.\beta + (1-g)(\dim X - 3) + n \tag{1.3}$$

and

$$[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}} \in H_{\mathrm{vdim}}(\overline{M}_{g,n}(X,\beta))).$$

The well-defined virtual dimension is called the *grading axiom*.

Before we go further, let's see what these invariants look like. As for $\overline{M}_{g,n}$, there is a universal curve over $\overline{M}_{g,n}(X,\beta)$:

$$\pi : \mathcal{C} \to \overline{M}_{g,n}(X,\beta),$$

which defines the ψ-classes $\psi_i, i = 1,\dots n$, as for the moduli of curves. The most general GWI can be written as

$$\int_{[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_i (\psi_i^{k_i} \,\mathrm{ev}_i^*(\alpha_i))\,\mathrm{st}^*(\Omega), \tag{1.4}$$

where $\Omega \in H^*(\overline{M}_{g,n})$.

Convention 1.6. (i) The above "integral" or pairing between cohomology and homology is defined to be zero if the total degree of cohomology is not equal to the virtual dimension.

(ii) When the stabilization morphism is not defined, one can set $\mathrm{st}^*(\Omega) = 1$.

However, sometimes we are only concerned with the case when $\Omega = 1$

$$\langle \tau_{k_1}(\alpha_1), \dots, \tau_{k_n}(\alpha_n)\rangle_{g,n,\beta} := \int_{[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_i (\psi_i^{k_i} \,\mathrm{ev}_i^*(\alpha_i)).$$

These are generally called *gravitational descendents*. When $k_i = 0$ for all i, they are called *primary invariants*. As you can easily guess, the descendents are the "descendents" of the primary fields. "Gravitation" is involved because ψ classes are the gravitational fields of the "topological gravity".

Proposition 1.7. *As a matter of fact, the set invariants in* (1.4) *can be reduced to a subset when all* $k_i = 0$*, and* $3g - 3 + n \geq 0$ *(when the stabilization morphism is defined).*

Assuming this proposition, we can view GWIs as *multi-linear* maps

$$I^X_{g,n}(\beta) : H^*(X)^{\otimes n} \to H^*(\overline{M}_{g,n}),$$

[2]Noted added: Due to a change of heart of one organizer, the construction of virtual fundamental classes was covered in these lectures. However, due to the time constraint, I will not be able to put that lecture into these notes.

which will be called *GW maps*. This is the approach taken by Kontsevich, Manin, Beherend etc.. Due to the symmetry of the marked points, $I_{g,n}$ is S_n-invariant up to a sign. When all the cohomology classes are *algebraic classes*, there will be no sign. *We will ignore the sign* for simplicity.

Phrased this way, the existence of virtual classes implies that the GW maps are constructed by correspondences via the virtual classes as kernels. This is called the *motivic axiom*. In other words, motivic axiom says that GWIs are constructed out of a virtual class.

Kontsevich–Manin [8] lists 9 axioms, so we have 7 more to go.

The *S_n-covariance axiom* says that permuting the marked points will not change the invariants (up to a sign, which is ignored!)

The *effectivity axiom* says that if β is not an effective curve class, then the corresponding GWI vanishes. This should be obvious as the corresponding moduli stack is empty.

The *fundamental class axiom* says that for the forgetful morphisms: $\mathrm{ft}_i : \overline{M}_{g,n+1}(X,\beta) \to \overline{M}_{g,n}(X,\beta)$, the virtual class pull-backs to virtual class

$$\mathrm{ft}_i^*([\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}} = [\overline{M}_{g,n+1}(X,\beta)]^{\mathrm{vir}}.$$

Another is the *mapping to a point axiom.* Suppose $\beta = 0$, then it should be easy to see that

$$\overline{M}_{g,n}(X,0) = \overline{M}_{g,n} \times X.$$

This is a smooth DM stack with dimension $3g-3+n+\dim X$. However the virtual dimension is $3g-3+n+(1-g)\dim X$. Therefore, the virtual class is *not* the fundamental class, even though it is smooth and fundamental class exists. In this case, there is an *obstruction bundle*

$$\mathbb{E}^\vee \otimes T_X \to \overline{M}_{g,n} \times X,$$

where $\mathbb{E}^\vee$ is the dual of the Hodge bundle, which as you all know has rank g. The virtual class is

$$[\overline{M}_{g,n}(X,0)]^{\mathrm{vir}} = c_{top}(\mathbb{E}^\vee \otimes T_X) \cap [\overline{M}_{g,n} \times X].$$

Note that the virtual class is the fundamental class when $g = 0$.

The next two axioms are the *splitting axiom* and *genus reduction axiom*, collectively called *gluing tails axiom*.

Let ϕ be one of the gluing morphisms for moduli of *curves*, parallel to those in (1.1) and (1.2). Let $\{T_\mu\}$ be a basis of $H^*(X)$ (as a vector space) and $g_{\mu\nu} := \int_X T_\mu T_\nu$ is the matrix of Poincaré pairing. Let $g^{\mu\nu}$ be the inverse matrix of $g_{\mu\nu}$. From Kunneth formula we know that the class of the diagonal in $H^*(X \times X)$ is $\sum_{\mu,\nu} g^{\mu\nu} T_\mu \otimes T_\nu$.

The axioms can be written as the following two equations:

$$\begin{aligned}\phi^*(I_{g,n,\beta}((\otimes_{i=1}^n)\alpha_i)) = &\sum_{\beta'+\beta''=\beta}\sum_{n'+n''=n}\sum_{\mu,\nu}\\ &I_{g_1,n'+1,\beta'}\left((\otimes_{i'=1}^{n'}\alpha_{i'})\otimes T_\mu\right) g^{\mu\nu} I_{g_2,n''+1,\beta''}\left((\otimes_{i''=n'+1}^{n}\alpha_{i''}\otimes T_\nu\right)\\ \phi^*(I_{g,n,\beta}((\otimes_{i=1}^n)\alpha_i)) = &\sum_{\mu,\nu} I_{g-1,n+2,\beta}\left((\otimes_{i=1}^n)\otimes T_\mu\otimes T_\nu\right) g^{\mu\nu}.\end{aligned}$$

Exercise 1.8. Rephrase these axioms in two different ways. First, in terms of the numerical invariants. (Easy!) Second, use GW maps, but without introducing the basis of $H^*(X)$. (So that it can be applied to Chow groups.)

In the remaining subsection, we will introduce the last axiom and show how to prove Proposition 1.7.

Exercise 1.9. Let $\bar{\psi}_i := \mathrm{st}^*(\psi_i)$ be the ψ-classes pulled-back from $\overline{M}_{g,n}$. Convince yourself that[3]

$$(\psi_i - \bar{\psi}_i)\cap[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}} = [D_i]^{\mathrm{vir}},$$

where D_i is the virtual divisor on $\overline{M}_{g,n}(X,\beta)$ defined by the image of the gluing morphism

$$\sum_{\beta'+\beta''=\beta}\overline{M}_{0,2}^{(i)}(X,\beta')\times_X\overline{M}_{0,n}(X,\beta'')\to\overline{M}_{g,n}(X,\beta),$$

where the i-th marked point goes to $\overline{M}_{0,2}^{(i)}(X,\beta')$ and the remaining $n-1$ points to $\overline{M}_{0,n}(X,\beta'')$.

With this exercise, we can see that one can exchange ψ-classes with $\bar{\psi}$-classes, which are pulled-back from $\overline{M}_{g,n}$, and the boundary divisors. The GWI associated with boundary divisors can be written as GWI of lower order classes, by the splitting and genus reduction axioms, and others. (Exercise: What is a good inductive order?)

To show Proposition 1.7, we still need to deal with the cases when $g=0$ and $n\le 2$, or $(g,n)=(1,0)$, for which the stabilization morphisms, and therefore the invariants, are not defined. We need the *divisor axiom*. Let D be a divisor on X and ft_{n+1} be the forgetful morphism, then

$$\begin{aligned}&\int_{[\overline{M}_{g,n+1}(X,\beta)]^{\mathrm{vir}}}\prod_{i=1}^n(\mathrm{ev}_i^*(\alpha_i)).\,\mathrm{st}^*\,\bar{\mathrm{ft}}_{n+1}^*(\Omega)\,.\,\mathrm{ev}_{n+1}^*([D])\\ =&(\int_\beta D)\int_{[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}}\prod_{i=1}^n(\mathrm{ev}_i^*(\alpha_i)).\,\mathrm{st}^*(\Omega).\end{aligned}$$

Here and elsewhere, $\bar{\mathrm{ft}}$ are the forgetful morphisms for moduli of curves.

[3]Some properties of virtual classes, which will be mentioned later, must be assumed. For the time being, assume that the moduli are all connected, smooth, projective variety with the correct dimension.

Exercise 1.10. (i) Convince yourself that the divisor axiom holds. You will need to use the fundamental class axiom. Then use the projection formula for ft_{n+1}.

(ii) Prove the divisor axiom for descendents

(1.5)

$$\begin{aligned}&\langle \alpha_1\psi_1^{k_1},\dots,\alpha_n\psi_n^{k_n}, D\rangle_{g,n+1,\beta,}\\ &=\int_\beta D\langle \alpha_1\psi_1^{k_1},\dots,\alpha_n\psi_n^{k_n}\rangle_{g,n,\beta,}+\sum_{i=1}^n\langle \alpha_1\psi_1^{k_1},\dots,D.\alpha_i\psi_i^{k_i-1},\dots,\alpha_n\psi_n^{k_n}\rangle_{g,n,\beta,},\end{aligned}$$

where by convention $\psi^{-1}:=0$. You will need the *virtual* version of the comparison theorem for ψ classes under forgetful morphisms, as for curves,

$$\psi_i=\mathrm{ft}_{n+1}^*(\psi_i)+E_i, \tag{1.6}$$

where

$$E_i=\overline{M}_{0,3}^{(i,n+1)}(X,0)\times_X\overline{M}_{g,n}(X,\beta)\subset\overline{M}_{g,n+1}(X,\beta),$$

where $i\le n$ and the i-th and $n+1$-st points lie in the first moduli factor.

(iii) Prove that the divisor axiom will uniquely determine those GWIs with $3g-3+n<0$.

Now, we have all the tools to prove Proposition 1.7. Remember what you learned about the comparisons of ψ-classes with respect to forgetful morphisms. They will be needed here.

Exercise 1.11. Prove Proposition 1.7.

Remark 1.12. The *enumerative interpretation* of GWIs is not always clear. Morally, the primary invariant $\langle\alpha_1,\dots,\alpha_n\rangle_{g,n,\beta}$ should count the number of n-pointed genus g curves in X with degree β, such that the i-th marked point lies in the i-th cycle, (the Poincaré dual of) α_i, all in general position. When genus is zero, and X is homogeneous, e.g. projective, the above interpretation actually holds.

2. Some GWT generating functions and their structures

2.1. **Generating functions.** It is often useful to form the generating functions of GWIs. Indeed, most of the structures in GWT only reveal themselves in terms of generating functions. As Fulton said, this is a gift from physics. The rest mathematicians *might* be able to figure out....

The first one is the genus g descendent potential. Recall the descendents look like

$$\langle\tau_{k_1}(\alpha_1),\dots,\tau_{k_n}(\alpha_n)\rangle_{g,n,\beta}:=\int_{[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}}\prod_i(\psi_i^{k_i}\,\mathrm{ev}_i^*(\alpha_i)).$$

At each marked point, the *insertion* can be $\alpha\otimes\psi^k$ for any k. Abstractly, we can think of the insertion come from an infinite dimensional vector space

$$\mathcal{H}_t:=\oplus_{k=0}^\infty H^*(X), \tag{2.1}$$

with basis $\{T_\mu \psi^k\}$, even though those vectors might have relations. (Think of this as the "universal" space, independent of β, n etc.. The actual spaces for insertions are quotients.) Let $\{t_k^\mu\}$ be the dual coordinates and

$$t := \sum_{\mu,k} t_k^\mu T_\mu \psi^k$$

be a general vector in $\mathcal{H}_t$. The *genus g descendent potential* is

$$F_g(t) := \sum_{n,\beta} \frac{q^\beta}{n!} \langle \otimes^n t \rangle_{g,n,\beta} := \sum_{n,\beta} \frac{q^\beta}{n!} \langle \underbrace{t, \dots, t}_{n \text{ insertions}} \rangle_{g,n,\beta}.$$

The variables $\{q^\beta\}_{\beta \in NE(X)}$ are called *Novikov variables*. Since $NE(X)$ is a cone, Novikov variables has a ring structure $q^{\beta_1} q^{\beta_2} = q^{\beta_1+\beta_2}$. It is called the *Novikov ring*.[4]

Convention 2.1. Denote Λ the Novikov ring of X. We will use $H^*(X)[[q]]$ to stand for $H^*(X, \Lambda)$.

If we want to replace the ψ classes by $\bar{\psi}$ classes, we will have to make sure that the target of the stabilization morphism exists. Let

$$s := \sum_{\mu,k} s^\mu T_\mu, \qquad \bar{t} := \sum_{\mu,k} \bar{t}_k^\mu T_\mu \bar{\psi}^k.$$

The *genus g ancestor potential* is defined as

$$\bar{F}_g(\bar{t}, s) := \sum_{l,m,\beta} \frac{q^\beta}{l!\, m!} \int_{[\overline{M}_{g,l+m}(X,\beta)]^{\text{vir}}} \bar{t}^{\otimes l} \otimes s^{\otimes m},$$

where the $\bar{\psi}$ classes are pullbacks from the composition of stabilization and forgetful morphisms:

$$\overline{M}_{g,l+m}(X, \beta) \to \overline{M}_{g,l+m} \to \overline{M}_{g,l}.$$

As remarked above, the indices of the summation, l, m, β, must ensure not only the existence of $\overline{M}_{g,m+l}(X, \beta)$, but also of $\overline{M}_{g,m}$. Who are the children of these ancestors, I often wonder?

2.2. **Quantum rings.** The splitting axiom at genus zero, combined with the permutation invariance of the invariants, give the *associativity* of the quantum rings, as we will proceed to show. Consider the generating function of genus zero primary invariants

$$F_0(s) := \sum_{n,\,\beta} \frac{q^\beta}{n!} \langle s^{\otimes n} \rangle_{0,n,\beta}.$$

[4]To be more precise, I will have to say that Novikov ring is the formal completion of the semigroup ring $\mathbb{C}[NE(X)]$ in the I-adic topology, where $I \subset \mathbb{C}[NE(X)]$ is the ideal generated by nonzero elements in $NE(X)$.

Define a product structure $*$ on $H^*(X)[[q]]$ by

$$T_\mu *_s T_\nu := \sum_{\delta,\epsilon} \left(\frac{\partial}{\partial s^\mu} \frac{\partial}{\partial s^\nu} \frac{\partial}{\partial s^\delta} F_0(s) \right) g^{\delta\epsilon} T_\epsilon.$$

Exercise 2.2. Show that $*_{q=0}$ gives the usual intersection/cup product.

Exercise 2.3. Show that the associativity of the product $*$ is equivalent to the following equation, often called WDVV equation after B. Dubrovin.

$$\begin{aligned} &\left(\frac{\partial}{\partial s^\mu} \frac{\partial}{\partial s^\nu} \frac{\partial}{\partial s^a} F_0(s) \right) g^{ab} \left(\frac{\partial}{\partial s^\epsilon} \frac{\partial}{\partial s^\delta} \frac{\partial}{\partial s^b} F_0(s) \right) \\ = &\left(\frac{\partial}{\partial s^\mu} \frac{\partial}{\partial s^\delta} \frac{\partial}{\partial s^a} F_0(s) \right) g^{ab} \left(\frac{\partial}{\partial s^\epsilon} \frac{\partial}{\partial s^\nu} \frac{\partial}{\partial s^b} F_0(s) \right). \end{aligned}$$

In other words, the function on the LHS is invariant under S_4 action.

To show the associativity of the $*$, one can use the following composition of stabilization and forgetful morphisms:

$$\overline{M}_{0,n\geq 4}(X,\beta) \to \overline{M}_{0,4} \cong \mathbb{P}^1.$$

Then notice that the LHS of the WDVV equation corresponds to a virtual boundary divisor in $\overline{M}_{0,n}(X,\beta)$, which is the pullback of one of the boundary point in $\overline{M}_{0,4}$ (capped with the virtual class), while the RHS correspond to another. Since the point class in $\mathbb{P}^1$ are rational/homological equivalent by definition, the invariants have to be equal, (*assuming* the axioms of the virtual classes).

Exercise 2.4. Check the above statements. What are the axioms one has to use?

This associative ring structure on $H^*(X)[[q]]$ is often called the *big quantum cohomology*, to distinguish itself from the *small quantum cohomology*, where the ring structure is defined by $*_{s=0}$, or better, by restricting s to the divisorial coordinates. The "equivalence" can be seen from the following exercise.

Exercise 2.5. Let $X = \mathbb{P}^r$, and let

$$s = s^0 1 + s^1 h + \ldots + s^r h^r,$$

and $s' := s|_{s^1=0}$. Verify that

$$F_g(s) = \sum_{d,n} \frac{q^{d\ell} e^{ds^1}}{n!} \langle s'^{\otimes n} \rangle_{0,n,d\ell}.$$

What about a general X?

2.3. **Computing GWIs I: WDVV.** Let $X = \mathbb{P}^2$. Let us proceed to find genus zero primary invariants

$$\langle \alpha_1, \dots, \alpha_n \rangle_{n,d} = \langle \alpha_1, \dots, \alpha_n \rangle_{g=0,n,d\ell}^X.$$

The insertion classes α_j can be $1, h, h^2 = pt$.

Exercise 2.6. (i) Show, by the fundamental class axiom, if any $\alpha_j = 1$, the invariant vanishes unless $(n, d) = (3, 0)$.

(ii) Show, by mapping to a point axiom, that if $d = 0$, the invariant vanishes unless $n = 3$. (In this case, quantum cohomology is classical cohomology.)

If $\alpha_j = h$, this can be taken cared of by divisor axiom, as in Exercise 2.5. So we only have to calculate GWIs of the following form

$$\langle pt^{\otimes^n} \rangle_{n,d}.$$

Now the virtual dimension, in this case equal to the actual dimension, is $3d + n - 1$. In order for the above invariants not to vanish a priori, we have to require the cohomological degree, which is $2n$, to be equal to the virtual dimension. That is, $n = 3d - 1$. Define

$$N_d := \langle pt^{\otimes^{3d-3}} \rangle_{n=3d-1,d}.$$

Exercise 2.7. Prove that WDVV equations in Exercise 2.3 give the following recursive equations for N_d:

$$N_d = \sum_{d_1+d_2=d, d_i>0} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right) \tag{2.2}$$

By Remark 1.12, we know that $N_1 = 1$, as there is exactly one line in X through 2 points in *general position*.

It is easy to see now N_d are completely determined by (2.2) and the initial condition $N_1 = 1$.

This sounds great, until we realize that there are very few cases which can be computed by WDVV alone. So let us look for something else.

2.4. **Equivariant cohomology and localization.** Suppose we have a $T := \mathbb{C}^*$ acting on X, a smooth projective variety. Then the universal T-bundle is[5]

$$ET := (\mathbb{C}^\infty \setminus \{0\}) \to BT := \mathbb{P}^\infty.$$

One can construct the associative X-bundle

$$X_T := X \times_T ET \to BT. \tag{2.3}$$

The equivariant cohomology of T is defined to be

$$H_T^*(X) := H^*(X_T).$$

[5]Actually, these are not algebraic schemes, so some kind of approximation is needed. It was all worked out carefully by B. Totaro and Edidin–Graham (and others).

In particular, if X is $\operatorname{Spec}\mathbb{C}$ with trivial T action,

$$H_T^*(\operatorname{Spec}\mathbb{C},\mathbb{Z}) = H^*(BT,\mathbb{Z}) \cong \mathbb{Z}[z].$$

By (2.3), $H^*(X_T)$ has a natural $\mathbb{Z}[t]$-module structure.

Given a (linearized) T-equivariant vector bundle $E \to X$, one can form the bundle $\pi_T : E_T \to X_T$. It is easy to see that at a fixed (geometric) point in BT, π_T is isomorphic to $E \to X$. One can define the equivariant chern classes to be

$$c_k^T(E) := c_k(E_T) \in H_T^*(X).$$

Next, we will need to know the localization. Let $j : X_i \hookrightarrow X$ be the fixed loci. Then the localization theorem(s) says that, for any $\alpha \in H^*(X)$,

$$\int_{[X]} \alpha = \sum_i \int_{[X_i]} \frac{i^*(\alpha)}{e(N_{X_i|X})}$$

where e_{N_i} is the $\mathbb{C}^*$-equivariant Euler class.

In fact, one can refine the above statement as follows.

Theorem 2.8 (Correspondence of residues)**.** *Suppose that $\mu : X \to Y$ is a $\mathbb{C}^*$-equivariant map. let $i : X_i \hookrightarrow X$ and $j : Y_j \hookrightarrow X$ be the fixed loci. Suppose furthermore that X_{i_k}'s are the only fixed components to map to Y_j. Then*

$$\sum_k (\mu|_{X_{i_k}})_* \left(\frac{i_k^*(\alpha)}{e(N_i)} \right) = \frac{j^*\mu_*(\alpha)}{e(N_j)}.$$

The usual localization is a summation of residues, with Y a point.

2.5. **Computing GWIs II: localization.** Arguably the most powerful technique in actual computation of GWIs is the *localization*. It is especially useful when combined with suitable generating functions. Unfortunately, that will in general require us to know some details of the construction of the virtual classes. Therefore, I will only talk about the simplest case. In fact, we won't even do a lot of localization for that matter.

Let $X = \mathbb{P}^r$. Define the *big J-function* of X to be a formal series in q and z^{-1}

$$J^{big} := \sum_d \sum_n \frac{q^d}{(n-1)!} (\mathrm{ev}_1) * \left(\frac{\mathrm{ev}_2^*(s) \dots \mathrm{ev}_n^*(s)}{z(z-\psi_1)} \right) \in H^*(X),$$

where

$$\frac{1}{z(z-\psi)} := \sum_k \frac{\psi^k}{z^{k+2}}.$$

Exercise 2.9. Prove that the data of all n-pointed GWIs, with at most one descendent insertion are packed in the big J-function.

Let us now restrict the variables from $s \in H^*(X)$ to $s^0 1 + s^1 h \in A^0 \oplus A^1$. The big J-function specializes to the *small J-function*

$$J^{small} = e^{(s^0 1 + s^1 h)/z} \left(1 + \sum_{d \geq 1} q^d e^{d s^1} \, \mathrm{ev}_*^d \frac{1}{z(z-\psi_1)} \right),$$

where $\mathrm{ev}^d : \overline{M}_{0,1}(X,d) \to X$ is the evaluation morphism. By the same understanding in Exercise 2.9, the small J-function is a generating function of all one-pointed descendents on X.

Exercise 2.10. (i) Prove the comparison theorem produces the following equation (*string equation*) of GWI:[6]

$$\langle \frac{\alpha_1}{(z-\psi_1)}, \alpha_2, \dots, \alpha_n, 1\rangle_{g,n+1,d} = \langle \frac{\alpha_1}{z(z-\psi_1)}, \alpha_2, \dots, \alpha_n\rangle_{g,n,d}$$

(ii) Use the string equation you just proved and the divisor equation for descendents (1.5) to show that the small J-function is of the above form.

Why J-function? There are more reasons than one. Firstly, it is often easier to compute, as we will see below. Secondly, when $H^*(X)$ is generated by divisor as in this case, there is a *reconstruction theorem* which reconstructs all n-pointed genus zero descendents.[7] Thirdly, it is intimately connected to the Frobenius structure which we will discuss in the next section.

To compute J-function for the projective spaces, one has to introduce the graph space

$$G_d := \overline{M}_{0,0}(X \times \mathbb{P}^1, (d,1)).$$

At this moment, we only need to know that G_d is a smooth DM stack with the correct dimension.

There is a "companion space" $\mathbb{P}^r_d$, which is defined to be the $(r+1)$-tuples of homogeneous degree d polynomials in two variables z_0, z_1, considered as the homogeneous coordinates on $\mathbb{P}^1$.

Exercise 2.11. (i) Show that $\mathbb{P}^r_d$ is isomorphic to $\mathbb{P}^{(r+1)(d+1)-1}$. Furthermore, it is naturally birational to G_d.

(ii) Let $\mathbb{C}^*$ acting on $\mathbb{P}^r_d$ by $(z_0 : z_1) \mapsto (z_0 : \eta z_1)$, with $\eta \in \mathbb{C}^*$. Show that the fixed loci are $(d+1)$ copies of $\mathbb{P}^r$.

Now, let $\mathbb{C}^*$ acts on $\mathbb{P}^1$, which induces an action on G_d. What are the fixed loci?[8]

Exercise 2.12. Convince yourself that there are $d+1$ components of the fixed point loci, and they are the images of $\overline{M}_{0,1}(X,d_1) \times_X \overline{M}_{0,1}(X,d-d_1)$ in G_d, for $d_1 = 0, \dots, d$. We use the convention that

$$\overline{M}_{0,1}(X,d) \times_X \overline{M}_{0,1}(X,0) := \overline{M}_{0,1}(X,d).$$

Now, I have to quote the result of Alexander Givental and Jun Li, which says that there is actually a $\mathbb{C}^*$-equivariant birational *morphism* $\mu : G_d \to \mathbb{P}^r_d$. It is easy to see that in this case there is only one N_{i_k}, which we denote

[6] It is also called the *puncture equation* (for topological gravity).

[7] I will not be able to tell you about the reconstruction. It can be found in [11].

[8] G_d is an orbifold, rather than a variety, but we will apply the localization here anyway, as it works! Actually, it has been proved to work is a more accurate description.

by N_i. Applying the correspondence of residues theorem to the case $d_1 = d$, we get

$$\begin{array}{ccc} G_d & \xrightarrow{\mu} & \mathbb{P}^r_d \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle j} \\ \overline{M}_{0,1}(\mathbb{P}^r, d) & \xrightarrow{\text{ev}} & \mathbb{P}^r. \end{array}$$

Exercise 2.13. Convince yourself that the lower horizontal morphism is actually the evaluation.

Now we will have to figure out the equivariant Euler classes of the normal bundles.

Exercise 2.14. (i) Show that $e(N_i) = z(z - \psi)$, where z is the *equivariant parameter*.

(ii) Show that $e(N_j) = \prod_{m=1}^{d}(h + mz)^{r+1}$.

Well, this exercise might be a little too hard if you have never seen a similar computation. So here are some hints. N_i is a rank two vector bundle, as you can see from the dimension counting. That means we have two-dimensional deformation space out of this fixed loci. The fixed locus consists of nodal rational curves with a rational branch mapping to $X \times \{0\}$ of degree $(d, 0)$, connected to another $\mathbb{P}^1$ mapping to $\{x\} \times \mathbb{P}^1$ of degree $(0, 1)$. The two deformation directions are: smoothing the node, and moving the image of the node away from the fixed point $\{0\} \in \mathbb{P}^1$. The first one gives the factor $(z - \psi)$ and the second z. As for (ii), it should be straightforward!

Now apply the correspondence of residues with $\alpha = 1$, we get

Theorem 2.15. *The small J-function of the projective spaces*

$$J^{\mathbb{P}^r} = e^{(s^0 1 + s^1 h)/z} \left(1 + \sum_{d=1}^{\infty} q^d \frac{1}{\prod_{m=1}^{d}(h + mz)^{r+1}}\right).$$

By Exercise 2.10, we now know all one-point descendents for projective spaces. If you are interested to compute the n-pointed descendents, you can get them by the reconstruction theorem in [11]. However, at this point I can't think of a good formulation of them, except in the $\mathbb{P}^1$ case, where one has the formulation by Okounkov–Pandharipande in their curve trilogy. There must be some integrable systems lurking behind.

3. Givental's axiomatic GWT at genus zero

The main references are [9, 10]. This section will only be used in Section 6.5. My guess is that it can be safely omitted without "serious consequence" for the remaining school week. Higher genus treatment is included as an appendix at the end.

3.1. **Axiomatic GWT.** our task is to distill the essence of the "geometric" GWT in an axiomatic framework, where no target variety is involved.

Let H be a vector space of dimension N with a distinguished element $\mathbf{1}$. Assume further that H is endowed with a nondegenerate symmetric bilinear form, or metric, $(\cdot,\cdot)$. One can think of them as $H = H^*(X)$ and $\mathbf{1} = 1 \in H^0(X)$. The bilinear form is the Poincaré intersection pairing.

Let $\{T_\mu\}$ be a basis of H and $T_{\mathbf{1}} = \mathbf{1}$. Let $\mathcal{H}$ denote the infinite dimensional vector space $H[z, z^{-1}]]$ consisting of Laurent polynomials with coefficients in H.[9] Introduce a symplectic form Ω on $\mathcal{H}$:

$$\Omega(f(z), g(z)) := \mathrm{Res}_{z=0}\langle f(-z), g(z)\rangle,$$

where the symbol $\mathrm{Res}_{z=0}$ means to take the residue at $z = 0$.

There is a natural polarization $\mathcal{H} = \mathcal{H}_q \oplus \mathcal{H}_p$ by the Lagrangian subspaces $\mathcal{H}_q := H[z]$ and $\mathcal{H}_p := z^{-1}H[[z^{-1}]]$ which provides a symplectic identification of $(\mathcal{H}, \Omega)$ with the cotangent bundle $T^*\mathcal{H}_q$ with the natural symplectic structure. $\mathcal{H}_q$ has a basis

$$\{T_\mu z^k\}, \quad 1 \le \mu \le N, \quad 0 \le k$$

with dual coordinates $\{q^k_\mu\}$. The corresponding basis for $\mathcal{H}_p$ is

$$\{T_\mu z^{-k-1}\}, \quad 1 \le \mu \le N, \quad 0 \le k$$

with dual coordinates $\{p^k_\mu\}$.

For example, if $\{T_i\}$ be an *orthonormal* basis of H.[10] An H-valued Laurent formal series can be written in this basis as

$$\ldots + (p^1_1, \ldots, p^N_1)\frac{1}{(-z)^2} + (p^1_0, \ldots, p^N_0)\frac{1}{(-z)} + (q^1_0, \ldots, q^N_0) + (q^1_1, \ldots, q^N_1)z + \ldots.$$

In fact, $\{p^i_k, q^i_k\}$ for $k = 0, 1, 2, \ldots$ and $i = 1, \ldots, N$ are the Darboux coordinates compatible with this polarization in the sense that

$$\Omega = \sum_{i,k} dp^i_k \wedge dq^i_k.$$

The parallel between $\mathcal{H}_q$ and $\mathcal{H}_t$, defined in (2.1), is evident. It is in fact given by the following affine coordinate transformation, called the *dilaton shift*,

$$t^\mu_k = q^\mu_k + \delta^{\mu\mathbf{1}}\delta_{k1}.$$

Definition 3.1. *Let $G_0(t)$ be a (formal) function on $\mathcal{H}_t$. The pair $(\mathcal{H}, G_0)$ is called a (polarized)* genus zero axiomatic theory *if G_0 satisfies three sets of genus zero tautological equations: the* Dilaton Equation (3.1)*, the* String Equation (3.2) *and the* Topological Recursion Relations *(TRR)* (3.3).

[9] Different completions of $\mathcal{H}$ are used in different places. This will be not be discussed details in the present article. See [10] for the details.

[10] The distinguished element $\mathbf{1}$ is not in this basis, unless $N = 1$.

$$\frac{\partial G_0(t)}{\partial t_1^{\mathbf{1}}}(t) = \sum_{k=0}^{\infty}\sum_{\mu} t_k^{\mu}\frac{\partial G_0(t)}{\partial t_k^{\mu}} - 2G_0(t), \tag{3.1}$$

$$\frac{\partial G_0(t)}{\partial t_0^{\mathbf{1}}} = \frac{1}{2}\langle t_0, t_0\rangle + \sum_{k=0}^{\infty}\sum_{\nu} t_{k+1}^{\nu}\frac{\partial G_0(t)}{\partial t_k^{\nu}}, \tag{3.2}$$

$$\frac{\partial^3 G_0(t)}{\partial t_{k+1}^{\alpha}\partial t_l^{\beta}\partial t_m^{\gamma}} = \sum_{\mu\nu}\frac{\partial^2 G_0(t)}{\partial t_k^{\alpha}\partial t_0^{\mu}}g^{\mu\nu}\frac{\partial^3 G_0(t)}{\partial t_0^{\nu}\partial t_l^{\beta}\partial t_m^{\gamma}}, \quad \forall \alpha,\beta,\gamma,k,l,m. \tag{3.3}$$

In the case of geometric theory, $G_0 = F_0^X$ It is well known that F_0^X satisfies the above three sets of equations (3.1) (3.2) (3.3).

Exercise 3.2. (i) Check that F_0^X satisfies these three equations. They are similar to the corresponding equations for moduli of curves.

(ii) Prove that the dilaton equation, when changed to the q-variables, is the same as the Euler equation of $G_0(q)$

$$\sum_{k=0}^{\infty}\sum_{\mu} q_k^{\mu}\frac{\partial G_0}{\partial q_k^{\mu}} = 2G_0(t).$$

It means that $G_0(q)$ is *homogeneous of degree two.*

3.2. **Overruled Lagrangian cones.** Givental [6] gives a beautiful geometric reformation of the polarized genus zero axiomatic theory in terms of *Lagrangian cones* in $\mathcal{H}$. The new formulation is independent of the polarization. That is, it is formulated in terms of the symplectic vector space $(\mathcal{H},\Omega)$, without having to specify a half dimensional space $\mathcal{H}_q$ such that $\mathcal{H} \cong T^*\mathcal{H}_q$.

The descendent Lagrangian cones are constructed in the following way. Denote by $\mathcal{L}$ the graph of the differential dG_0:

$$\mathcal{L} = \{(p,q) \in T^*\mathcal{H}_q : \ p_k^{\mu} = \frac{\partial}{\partial q_k^{\mu}}G_0\} \subset T^*\mathcal{H}_q.$$

[11] $\mathcal{L}$ is therefore considered as a formal germ of a Lagrangian submanifold in the space $(\mathcal{H},\Omega)$.

Theorem 3.3. *$(\mathcal{H}, G_0)$ defines a polarized genus zero axiomatic theory if the corresponding Lagrangian cone $\mathcal{L} \subset \mathcal{H}$ satisfies the following properties: $\mathcal{L}$ is a Lagrangian cone with the vertex at the origin of q such that its tangent spaces L are tangent to $\mathcal{L}$ exactly along zL.*

A Lagrangian cone with the above property is also called *overruled* (descendent) Lagrangian cones.

[11] One might have to consider it as a formal germ at $q = -z$ (i.e. $t = 0$) of a Lagrangian section of the cotangent bundle $T^*\mathcal{H}_q = \mathcal{H}$ in the geometric theory, due to the convergence issues of G_0.

Remark 3.4. In the geometric theory, $F_0^X(t)$ is usually a formal function in t. Therefore, the corresponding function in q would be formal at $q = -\mathbf{1}z$. Furthermore, the Novikov rings are usually needed to ensure the well-definedness of $F_0^X(t)$.

3.3. **Twisted loop groups.** The main advantage of viewing the genus zero theory through this formulation is to replace $\mathcal{H}_t$ by $\mathcal{H}$ where a symplectic structure is available and the polarization becomes inessential. Therefore many properties can be reformulated in terms of the symplectic structure Ω and hence independent of the choice of the polarization. This suggests that the space of genus zero axiomatic Gromov–Witten theories, i.e. the space of functions G_0 satisfying the string equation, dilaton equation and TRRs, has a huge symmetry group.

Definition 3.5. *Let $L^{(2)}GL(H)$ denote the* twisted loop group *which consists of* $\mathrm{End}(H)$*-valued formal Laurent series $M(z)$ in the indeterminate z^{-1} satisfying $M^*(-z)M(z) = \mathbf{I}$. Here * denotes the adjoint with respect to $(\cdot,\cdot)$.*

The condition $M^*(-z)M(z) = \mathbf{I}$ means that $M(z)$ is a symplectic transformation on $\mathcal{H}$.

Theorem 3.6. [6] *The twisted loop group acts on the space of axiomatic genus zero theories. Furthermore, the action is transitive on the semisimple theories of a fixed rank N.*

When viewed in the Lagrangian cone formulation, Theorem 3.6 becomes transparent and a proof is almost immediate.

What is a semisimple theory? I won't really define it here, but just say that the (geometric) quantum cohomology algebra is called semisimple if it is diagonalizable with nonzero eigenvalues. Since our quantum products are all commutative, it is equivalent to saying that no element is nilpotent.

3.4. **Saito (or Frobenius) manifolds.** Well, it seems unavoidable that I have to give a definition of the *Frobenius manifolds*.[12] The notion was introduced by B. Dubrovin.

Definition 3.7. *An (even) complex* Saito *(or Frobenius)* manifold *consists of four mathematical structures $(H, g, A, \mathbf{1})$:*

- *H is a complex manifold of dimension N.*
- *$g := (\cdot,\cdot)$ is a holomorphic, symmetric, non-degenerate bilinear form on the complex tangent bundle TH.*
- *A is a holomorphic symmetric tensor*

$$A : TH \otimes TH \otimes TH \to \mathcal{O}_H.$$

[12]This might be another misnomer, probably worse than the term "Hilbert schemes". As everybody knows, Hilbert schemes should have been called "Grothendieck scheme", and it was probably Grothendieck's self-effacing personality which contributed to establishing the term. The so-called Frobenius manifolds should have been called the "(Kyoji) Saito manifolds", in the same way P. Deligne coined the term "Shimura varieties".

- **1** *is a holomorphic vector field on* H.

A and g together define a commutative product $*$ on TH by

$$(X * Y, Z) := A(X, Y, Z),$$

where X, Y, Z are holomorphic vector fields.

The above quadruple satisfies the following conditions:

(1) Flatness: *g is a flat holomorphic metric.*
(2) Potential: *H is covered by open sets U, each equipped with a commuting basis of g-flat holomorphic vector fields X_i, $i = 1, \dots, N$, and a holomorphic potential function Φ on U such that*

$$A(X_i, X_j, X_k) = X_i X_j X_k(\Phi).$$

(3) Associativity: *$*$ is an associative product.*
(4) Unit: *The identity* **1** *is a g-flat unit vector field.*

In geometric GWT, $H = H^*(X)$ and $*$ is the quantum product. The (non-equivariant) GWT carries additional information, the grading and divisor axiom. Neither Givental's nor Dubrovin's formulation includes the divisor axiom and it makes sense. For example, if one wishes to think about generalized cohomology, like K-theory, the divisor axiom has to be the first to go.

As for the grading, it can be translated to the *conformal structure* for Saito manifolds. Since everything is holomorphic, we will omit the adjective.

Definition 3.8. *Let E be a vector field on H and $\mathcal{L}_E$ be the Lie derivative. E is called the* Euler vector field *if the following three conditions are satisfied:*

(1) *$\mathcal{L}_E(g) = (2 - D)g$ for a constant D,*
(2) *$\mathcal{L}_E(*) = r*$ for a constant r,*
(3) *$\mathcal{L}_E(\mathbb{1}) = v\mathbb{1}$ for a constant v.*

A conformal Saito manifold *is a Saito manifold equipped with an Euler vector field.*

Remark 3.9. One can define a family of *flat connection*

$$\nabla_{z,X}(Y) := \nabla_X(Y) - \frac{1}{z} X * Y,$$

where ∇ is the Levi-Civita connection of the metric g. The fundamental solution matrix of $\nabla_{z,X}$ is an $N \times N$ matrix, with (independent) solutions as *column* vector. Then the *row* vector in the component of **1** is the *big J-function*. If these are defined in the geometric setting, where the divisor axiom holds, then the small J-function is the restriction of the big J-function to the divisorial directions $H^1(X)$. These will be discussed in the next subsection.

3.5. **Relation between axiomatic and the Saito structures.** Consider the intersection of the cone $\mathcal{L}$ with the affine space $-z + z\mathcal{H}_p$. The intersection is parameterized by $\tau \in H$ via its projection to $-z + H$ along $\mathcal{H}_p$ and can be considered as the graph of a function from H to $\mathcal{H}$ called the *J-function*:

$$\tau \mapsto zJ(-z,\tau) = -z + \tau + \sum_{k>0} J_k(\tau)(-z)^{-k}.$$

Lemma 3.10. *J-functions satisfy the following two properties.*

(1) *$\partial J/\partial\tau^\delta$ form a fundamental solution of the pencil of flat connections depending linearly on z^{-1}:*

$$z\frac{\partial}{\partial\tau^\lambda}\left(\frac{\partial J}{\partial\tau^\delta}\right) = \sum_\mu A^\mu_{\delta\lambda}(\tau)\left(\frac{\partial J}{\partial\tau^\mu}\right). \tag{3.4}$$

(2) *$z\partial J/\partial\tau^1 = J$. Equivalently $(A^\mu_{1\lambda})$ is the identity matrix.*

Corollary 3.11. (1) $A^\mu_{\delta\lambda} = A^\mu_{\lambda\delta}$.

(2) *The multiplications on the tangent spaces $T_\tau H = H$ given by*

$$\phi_\delta * \phi_\lambda = \sum_\mu A^\mu_{\delta\lambda}(\tau)\phi_\mu$$

define associative commutative algebra with the unit 1.

The above $*$-multiplication and the inherited inner product on H satisfies the *Frobenius property*

$$(a * b, c) = (a, b * c)$$

and satisfy the integrability (zero curvature) condition imposed by (3.4). This is what a Saito structure is defined earlier. Note that the above formulation is equivalent to Dubrovin's original definition: One can use the generating function F_0 to show that in fact $A^\mu_{\delta\lambda} = \sum g^{\mu\nu}\,\langle\phi_\nu, \phi_\delta, \phi_\lambda\rangle(\tau)$ and therefore $F_0(\tau, 0, 0, ...)$ satisfies the WDVV-equation.

Conversely, given a Saito structure one recovers a J-function by looking for a fundamental solution matrix to the system

$$z\frac{\partial}{\partial\tau^\lambda}S = \phi_\lambda * S$$

in the form of an operator-valued series

$$S = 1 + S_1(\tau)z^{-1} + S_2(\tau)z^{-2} + ...$$

satisfying

$$S^*(-z)S(z) = 1$$

Such a solution always exists and yields the corresponding J-function $J^\delta(z,\tau) = [S^*(z,\tau)]^\delta_1$ and the Lagrangian cone $\mathcal{L}$ enveloping the family of Lagrangian spaces $L = S^{-1}(z,\tau)\mathcal{H}_q$ and satisfies the Properties in Theorem 3.3 A choice of the fundamental solution S is called a *calibration* of the Saito structure. The calibration is unique up to the right multiplication $S \mapsto SM$ by elements $M = 1 + M_1 z^{-1} + M_2 z^{-2} + ...$ of the "lower-triangular" subgroup

in the twisted loop group. Thus the action of this subgroup on our class of cones $\mathcal{L}$ changes calibrations but does not change Saito Structures (while more general elements of $L^{(2)}GL(H)$, generally speaking, change Saito structures as well).

Summarizing the above, one has

Theorem 3.12. *Given a Lagrangian cone satisfying conditions in Theorem 3.3 is equivalent to given a (formal) germ of Saito manifold.*

4. Relative GWI and degeneration formula

A good reference, I was told, is Jun Li's ICTP notes [12].

4.1. **Relative GWT.** Let Y be a smooth projective variety and E a smooth divisor in Y. In order to define the relative stable morphisms, some notations must be introduced.

Let $P_1 := \mathbb{P}_E(N_{E|Y} \oplus \mathscr{O}_E)$ be the $\mathbb{P}^1$-bundle over E. The bundle $P_1 \to E$ has two disjoint sections. One is $\mathbb{P}_E(N_{E|Y})$ and the other is $\mathbb{P}_E(\mathscr{O}_E)$. The former has normal bundle $N^\vee$, and is called the zero section. The latter has normal bundle N, and is called the infinity section.

By gluing l copies of P_1, with the infinity section of the j-th copy glued to the zero section of $(j+1)$-st copy, we can form P_l, a *singular* variety. Call the zero/infinity section of P_l to be the zero/infinity section of the first/last copy of the $\mathbb{P}^1$-bundle. They are denoted E_0 and E_∞ respectively.

Define Y_l by gluing Y with P_l on E and E_0. Define the automorphism of Y_l to be

$$\mathrm{Aut}(Y_l) := (\mathbb{C}^*)^l,$$

with $\mathbb{C}^*$ acting on the $\mathbb{P}^1$-fibers.

The "topological type" of the relative stable morphisms are encoded in the following data:

$$\Gamma = (g, n, \beta, \mu)$$

with $\mu = (\mu_1, \dots, \mu_\rho) \in \mathbb{N}^\rho$ a partition of the intersection number

$$(\beta.E) = |\mu| := \sum_{i=1}^{\rho} \mu_i.$$

In the language of combinatoric, $|\mu|$ is the *size* of μ and ρ the *length*. The corresponding moduli stack, which we will define now, is denoted $\overline{M}_\Gamma(Y, E)$.

The geometric points of this stack correspond to morphisms

$$C \xrightarrow{f} Y_l \to Y,$$

where C is an (n, ρ)-pointed, genus g prestable curve such that

$$f^*(E_\infty) = \sum_{i=1}^{\rho} \mu_i y_i.$$

Here the first n ordered points are denoted x_i and the last *unordered* ρ points are denoted y_i. We will call x_i the *ordinary* (or interior) marked points, and y_i the *relative* marked points. In addition, f must satisfy a *predeformability* condition:

- The preimage of singular locus of Y_l must lie on the nodes of C;
- for any such node p, the two branches of C at p map to different irreducible components of Y_l, and they have the same contact orders with the singular divisor at p.

Two such relative morphisms $f : C \to Y_l$ and $f' : C' \to Y_l$ are called *isomorphic*, if there are isomorphisms of pointed curves $g : C \to C'$ and $h \in \mathrm{Aut}(Y_l)$, such that $f' = h \circ f \circ g$. A relative morphism is called *stable* if there is no infinitesimal automorphism.

Generalizing the above setting to the families, one gets a moduli functor and hence the corresponding moduli stack of stable relative morphisms, which is what we denoted $\overline{M}_\Gamma(Y, E)$. I assume that you will find this generalization more or less obvious except possibly the predeformability condition. Let

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{f} & \mathcal{X} & \longrightarrow & X \times S \\ {\scriptstyle\pi}\downarrow & & & & \\ S & & & & \end{array}$$

be a family of relative morphisms over S. Denote $\mathrm{Spec}(A)$ the (étale/formal) neighborhood of a point $s \in S$, then

$$\mathcal{C}|_{\mathrm{Spec}(A)} \cong \mathrm{Spec}\left(\frac{A[u,v]}{(uv-\lambda_1)}\right), \quad \mathcal{X}|_{\mathrm{Spec}(A)} \cong \mathrm{Spec}\left(\frac{A[x,y,z_1,z_2,...]}{(xy-\lambda_2)}\right).$$

The map f in these local coordinates is of the following form:

$$x \mapsto \eta_1 u^m,\ y \mapsto \eta_2 v^m,$$

with η_i units in $A[u,v]/(uv-\lambda_1)$ and no restriction on z_i.

Now, we will take this on faith, as we have always been anyway, that there is a well-defined *virtual class* for each $\overline{M}_\Gamma(Y, E)$. For $A \in H^*(Y)^{\otimes n}$ and $\varepsilon \in H^*(E)^{\otimes \rho}$, the relative invariant of stable maps with topological type Γ (i.e. with contact order μ_i in E at the i-th contact point) is

$$\langle A \mid \varepsilon, \mu \rangle_\Gamma^{(Y,E)} := \int_{[\overline{M}_\Gamma(Y,E)]^{virt}} \mathrm{ev}_Y^* A \cup \mathrm{ev}_E^* \varepsilon$$

where $\mathrm{ev}_Y : \overline{M}_\Gamma(Y, E) \to Y^n$, $\mathrm{ev}_E : \overline{M}_\Gamma(Y, E) \to E^\rho$ are evaluation morphisms on marked points and contact points respectively.

For relative GWT, the slight generalization to invariants with disconnected domain curves, which we distinguish by $\bullet$ on the upper right corner, is necessary for applications. If $\Gamma = \coprod_\pi \Gamma^\pi$, the relative invariants

$$\langle A \mid \varepsilon, \mu \rangle_\Gamma^{\bullet(Y,E)} := \prod\nolimits_\pi \langle A \mid \varepsilon, \mu \rangle_{\Gamma^\pi}^{(Y,E)}$$

are defined to be the product of the connected components.

Remark 4.1. Some authors prefer to use the *ordered* ρ points. The moduli there become finite covers of ours, and the differences can be easily accounted for by combinatorial factors.

4.2. **Degeneration formula.** Even though we might not a priori be interested in the relative invariants, the following situation automatically leads us to them.

Let $\pi : W \to \mathbb{A}^1$ be a double point degeneration. That is, π is a flat family, W_t are smooth for $t \neq 0$,[13] and the central fiber W_0 is a union $A \cup B$ with A and B smooth, intersecting transversally. We know that GWT of W_t are all isomorphic, as GWIs are deformation invariant. So in some sense GWT of W_0 "must" be equal to GWT of W_t. This is when the degeneration formula applies.

Let us describe the degeneration formula in the case of the *deformation to the normal cone*.[14] Let X be a smooth variety and $Z \subset X$ be a smooth subvariety. Let $W \to \mathbb{A}^1$ be its deformation to the normal cone.. That is, W is the blow-up of $X \times \mathbb{A}^1$ along $Z \times \{0\}$. Denote $t \in \mathbb{A}^1$ the deformation parameter. Then $W_t \cong X$ for all $t \neq 0$ and $W_0 = Y_1 \cup Y_2$ with

$$\phi = \Phi|_{Y_1} : Y_1 \to X$$

the blow-up along Z and

$$p = \Phi|_{Y_2} : Y_2 := \mathbb{P}_Z(N_{Z/X} \oplus \mathscr{O}) \to Z \subset X$$

the projective completion of the normal bundle. $Y_1 \cap Y_2 =: E = \mathbb{P}_Z(N_{Z/X})$ is the ϕ-exceptional divisor which consists of "the infinity part" of the projective bundle $\mathbb{P}_Z(N_{Z/X} \oplus \mathscr{O})$.

Since the family $W \to \mathbb{A}^1$ is a degeneration of a trivial family, all cohomology classes $\alpha \in H^*(X, \mathbb{Z})^{\oplus n}$ have global liftings and the restriction $\alpha(t)$ on W_t is defined for all t. Let $j_i : Y_i \hookrightarrow W_0$ be the inclusion maps for $i = 1, 2$. Let $\{e_i\}$ be a basis of $H^*(E)$ with $\{e^i\}$ its dual basis. $\{e_I\}$ forms a basis of $H^*(E^\rho)$ with dual basis $\{e^I\}$ where $|I| = \rho$, $e_I = e_{i_1} \otimes \cdots \otimes e_{i_\rho}$. The *degeneration formula* expresses the absolute invariants of X in terms of the relative invariants of the two smooth pairs (Y_1, E) and (Y_2, E):

$$\langle \alpha \rangle^X_{g,n,\beta} = \sum_I \sum_{\eta \in \Omega_\beta} C_\eta \left\langle j_1^* \alpha(0) \,\middle|\, e_I, \mu \right\rangle^{\bullet(Y_1,E)}_{\Gamma_1} \left\langle j_2^* \alpha(0) \,\middle|\, e^I, \mu \right\rangle^{\bullet(Y_2,E)}_{\Gamma_2}. \tag{4.1}$$

Here $\eta = (\Gamma_1, \Gamma_2, I_\rho)$ is an *admissible triple* which consists of (possibly disconnected) topological types

$$\Gamma_i = \coprod_{\pi=1}^{|\Gamma_i|} \Gamma_i^\pi$$

with the same partition μ of contact order under the identification I_ρ of contact points. The gluing $\Gamma_1 +_{I_\rho} \Gamma_2$ has type (g, n, β) and is connected. In

[13] One can shrink $\mathbb{A}^1$ if necessary.

[14] Or, degeneration to the normal cone, if you prefer.

particular, $\rho = 0$ if and only if that one of the Γ_i is empty. The total genus g_i, total number of marked points n_i and the total degree $\beta_i \in NE(Y_i)$ satisfy the splitting relations

$$\begin{aligned} g &= g_1 + g_2 + \rho + 1 - |\Gamma_1| - |\Gamma_2|, \\ n &= n_1 + n_2, \\ \beta &= \phi_*\beta_1 + p_*\beta_2. \end{aligned}$$

The constants $C_\eta = m(\mu)/|\mathrm{Aut}\,\eta|$, where $m(\mu) = \prod \mu_i$ and $\mathrm{Aut}\,\eta = \{\, \sigma \in S_\rho \mid \eta^\sigma = \eta \,\}$. (When a map is decomposed into two parts, an (extra) ordering to the contact points is assigned. The automorphism of the decomposed curves will also introduce an extra factor. These contribute to $\mathrm{Aut}\,\eta$.) We denote by Ω the set of equivalence classes of all admissible triples; by Ω_β and Ω_μ the subset with fixed degree β and fixed contact order μ respectively.

In general, the degeneration formula applies to the more general setting of double point degeneration.[15] I trust that you can work out the form of the general degeneration formula, which is very similar to (4.1). However, in the general case, the cohomology classes in a fiber might not lift to the family, so the "families of classes" $\alpha(t)$ have to be part of the initial data. We will not use it explicitly and won't say anything more. The interested parties can go ahead and read it in [12].

Question 4.2. *Can one generalize Givental's axiomatic framework to the relative setting, and furthermore incorporate the degeneration formula?*[16]

5. Orbifolds and Orbifold GWT

A good reference for this section is [1]. By *orbifolds*, we mean smooth separated Deligne–Mumford stacks of finite type over $\mathbb{C}$.

5.1. **Chen–Ruan cohomology.** Let X be an orbifold. Its inertia stack IX is the fiber product

$$\begin{array}{ccc} IX & \longrightarrow & X \\ \downarrow & & \Delta\downarrow \\ X & \xrightarrow{\Delta} & X \times X, \end{array}$$

where Δ is the diagonal morphism. One can think of a point of IX as a pair (x, g), where $x \in X$ and $g \in \mathrm{Aut}_x(X)$. There is an involution

$$\mathbf{I} : IX \to IX$$

which sends (x, g) to (x, g^{-1}).

[15] This has recently been generalized to even more general situation using log geometry.

[16] I have posted this question to many, but never to a group of eager students. So I am hopeful....

The orbifold cohomology group of X is defined to be the cohomology group of IX:

$$H^*_{CR}(X) := H^*(IX).$$

For each component X_j of IX, we can assign a rational number, called the *age*. Pick a geometric point (x, g) in the component. g acts on vector space T_xX cyclically and decompose it into eigenspaces V_j with eigenvalues $\exp(i2\pi\frac{j}{r})$. The age of the component is defined to be $\sum_j \frac{j}{r} \dim V_j$. The *Chen–Ruan grading* of the $H^*_{CR}(X)$ is shifted by the age. That is, if $\alpha \in H^k(X_j) \subset H^*(IX)$, then

$$\deg_{CR}(\alpha) := k + \mathrm{age}(X_j).$$

The Poincaré pairing is defined by

$$\alpha_1 \otimes \alpha_2 \mapsto \int_{IX} \alpha_1 \cup I^*(\alpha_2).$$

Note that the pairing vanishes unless $\sum_j \deg_{CR}(\alpha_j) = \dim X$.

Example 5.1. Weighted projective stack $\mathbb{P}^w$ is the stack quotient

$$\mathbb{P}^w := \left[\left(\mathbb{A}^{r+1} \setminus 0\right) / \mathbb{C}^*\right],$$

where $\mathbb{C}^*$ acts with weights $(-w_0, -w_1, \ldots, -w_r)$. Components of the inertia stack are indexed by

$$F = \{\frac{k}{w_j} | 0 \le k < w_j \text{ and } 0 \le j \le r\},$$

such that

$$I\mathbb{P}^w = \amalg_{f \in F} \mathbb{P}(V^f)$$

with

$$V^f := \{(z_0, \ldots, z_r) \in \mathbb{A}^{r+1} | z_j = 0 \text{ unless } w_j f \in \mathbb{Z}\}$$
$$\mathbb{P}(V^f) = \left[\left(V^f \setminus 0\right) / \mathbb{C}^*\right],$$

such that $\mathbb{P}(V^f)$ is the locus of points of $\mathbb{P}^w$ with isotropy group containing $e^{i2\pi f}$.

The involution I maps the component $\mathbb{P}(V^f)$ to the component $\mathbb{P}(V^{\langle -f \rangle})$, where

$$\langle -f \rangle := (-f) - \lfloor (-f) \rfloor$$

is the fractional part of $-f$.

Exercise 5.2. Let $X = [Y/G]$ be a global quotient of a smooth projective variety Y by a finite group G. Show that

$$IX = \amalg_{(g)} [Y^g \,/\, C(g)],$$

where the disjoint union runs over conjugacy classes and $C(g)$ denotes the centralizer of g.

5.2. **orbifold GWT.** I will have to be even more evasive here. The moduli for stable morphisms to orbifolds and their virtual classes exist and enjoy similar properties (axioms) of GWT. The only significant revisions are:

1. The domain curves allow stacky structures as well. They are call *twisted curves*. Basically, an n-pointed twisted curve is a connected one dimensional orbifold such that

- its coarse moduli scheme is an n-pointed prestable curve;
- its stacky structures only happen at the marked points and at the nodes;
- those stacky structures are cyclic quotient, étale locally like $[\mathbb{A}^1/\mu_r]$;
- it has *balanced* cyclic quotient stack structures at nodes, étale locally like
$$\left[\operatorname{Spec}\frac{\mathbb{C}[x,y]}{(xy)}/\mu_r\right],$$
where $\zeta \in \mu_r$ acts as
$$\zeta : (x,y) \mapsto (\zeta x, \zeta^{-1}y).$$

2. An n-pointed *twisted stable map* f must be *representable*. Roughly, f sends automorphisms of x to those of $f(x)$, and representability means that it is injective. Naturally, we require that the induced morphism $\bar{f}$ at the level of coarse moduli schemes is stable in the sense defined earlier.

3. The definition of the evaluation morphisms also need some adjustments. Given a twisted stable map f, each marked point x_i determines a geometric point $(f(x_i), g)$, where f is defined as follows/ Near x_i the curve is isomorphic to $[\mathbb{A}^1/\mu_r]$. Since f is representable, it determines an injective homomorphism $\mu_r \to \operatorname{Aut}_{f(x_i)}$. Since we work over $\mathbb{C}$, μ_r has a preferred generator $\zeta = e^{i2\pi/r}$. g is then the image of this generator under the above injective homomorphism. Therefore, it seems that we have a well-defined evaluation morphism to the inertia stack from each marked point. However, the above definition doesn't work in families! There are ways around it, fortunately, as were explained by Abramovich–Graber–Vistoli. For our purpose, we will pretend that the above evaluation morphisms actually work.

4. In general the moduli are disconnected and the virtual dimension of the components are different. On the substack of twisted stable morphisms where the evaluation of the j-th marked point maps X_j, the virtual dimension
$$\operatorname{vdim}(\overline{M}_{g,n}(X,\beta)) := -K_X.\beta + (1-g)(\dim X - 3) + n - \sum_j \operatorname{age}(X_j).$$

With the above revisions, one can then define the orbifold GWT as before.

Exercise 5.3. Check that a similar grading axiom holds for orbifold GWT by the virtual dimension formula given above.

Similarly, we can define the quantum rings for orbifolds. I have coyly avoided talking about the "classical cup product" in this case, as the "right" definition of that is to use the degree zero restriction of the quantum ring. Cf. Exercise 2.2. The latter is called *Chen–Ruan product*.

Remark 5.4. We don't have nearly as well-developed tools in computing orbifold GWIs. All the tools in the scheme case are applicable, but they are not nearly as powerful. Check out however the progress by Corti et. al..

6. Crepant transformation conjecture

The main references are [2, 3].

6.1. **K-equivalence aka crepant transformation.** Let X and X' be two smooth varieties, or Deligne–Mumford stacks (orbifolds). We say they are *K-equivalent*[17] if there are birational morphisms $\phi : Y \to X$ and $\phi' : Y \to X'$ such that $\phi^* K_X \sim \phi^* K_X$ (linearly equivalence of divisors). We also say they are crepant equivalent, or the birational transformation from one to the other *crepant transformation*.[18]

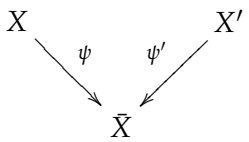
Why is K-equivalence relevant to GWT? Let us consider the functoriality of GWI. Since the cohomology theories are supposed to be functorial with respect to pullbacks, one can ask: "Is quantum cohomology functorial with respect to pullbacks?" Let us see how the functoriality can possibly hold.

Let $\phi : Y \to X$ be a morphism. To get GWIs, we need $\alpha_i \in H^*(X)$ and $\beta \in NE(Y)$. Then one can check whether it is possible at all that

$$\langle \otimes \phi^* \alpha_i \rangle_{g,n,\beta}^{Y} \stackrel{?}{=} \langle \otimes \alpha_i \rangle_{g,n,\phi_*\beta}^{X} ?$$

This can only happen if the virtual dimension of $\overline{M}_{g,n}(Y, \beta)$ is the same as $\overline{M}_{g,n}(X, \phi_*\beta)$, since the cohomology degrees are the same. Then a quick look into (1.3) should tell us that, in general, it implies that $\dim Y = \dim X$ and $K_Y.\beta = K_X.\phi_*\beta$. This leads us "naturally" to K-equivalence (crepant transformation). The statement of functoriality for the K-equivalence is/will be known as the *crepant transformation conjecture*, which we will explain.

But, you remember K-equivalence will never happen between birational morphisms (blowing-ups) of smooth varieties, following from the blowing-up formula for canonical divisors. See e.g. Hartshorne. However, it does happen for the birational maps, or transformations, $X \dashrightarrow X'$. The most common cases of K-equivalence have the following diagram



[17] The term was introduced by C.L. Wang and independently by V. Batyrev.

[18] I was told that the word "crepant" was introduced by M. Reid as "no dis-crepancy".

which include the *flops* and *crepant resolutions*. I will call it *K-equivalence of flopping type*, until I hear a better suggestion. In fact, for general K-equivalence, I don't know if CTC should hold. For K-equivalence of flopping type, there is a much better chance that CTC will hold as we will see soon. In any case, all proven (nontrivial) examples of CTC are of this type.

6.2. **Flops.** Let me first say what a flop is. ψ must be a *flopping contraction*, which means that it is proper, birational, *small* in the sense of Mori (i.e. the exceptional locus has codimension at least two in X) such that $\bar{X}$ a normal variety, and K_X is numerically ψ-trivial.

We will only be discussing the *ordinary* $\mathbb{P}^r$ *flops*. Since we require both X and X' to be smooth, this class of flops seem to provide fertile ground for testing ideas. First, let us understand a little better of the geometry of ordinary flops.

Recall that $\psi : X \to \bar{X}$ is the flopping contraction. Let $\bar{\psi} : Z \to S$ be the restriction map on the exceptional loci. Assume that

(i) $\bar{\psi}$ equips Z with a $\mathbb{P}^r$-bundle structure $\bar{\psi} : Z = \mathbb{P}_S(F) \to S$ for some rank $r+1$ vector bundle F over a smooth base S,
(ii) $N_{Z/X}|_{Z_s} \cong \mathscr{O}_{\mathbb{P}^r}(-1)^{\oplus(r+1)}$ for each $\bar{\psi}$-fiber Z_s, $s \in S$.

It is not hard to see that the corresponding *ordinary* $\mathbb{P}^r$ *flop* $f : X \dashrightarrow X'$ exists under the above two conditions. It can be constructed by first blowing up Z in X. That is, $\phi : Y := \mathrm{BL}_Z X \to X$. The exceptional divisor for ϕ is E a $\mathbb{P}^r \times \mathbb{P}^r$-bundle over S. Then one blows down E along another fiber direction $\phi' : Y \to X'$, with exceptional loci $\bar{\psi}' : Z' = \mathbb{P}_S(F') \to S$ for F' another rank $r+1$ vector bundle over S and also $N_{Z'/X'}|_{\bar{\psi}'-\text{fiber}} \cong \mathscr{O}_{\mathbb{P}^r}(-1)^{\oplus(r+1)}$.[19]

We start with the following elementary lemma.

Lemma 6.1. *Let $p : Z = \mathbb{P}_S(F) \to S$ be a projective bundle over S and $V \to Z$ a vector bundle such that $V|_{p^{-1}(s)}$ is trivial for every $s \in S$. Then $V \cong p^*F'$ for some vector bundle F' over S.*

Proof. Recall that $H^i(\mathbb{P}^r, \mathscr{O})$ is zero for $i \neq 0$ and $H^0(\mathbb{P}^r, \mathscr{O}) \cong \mathbb{C}$. By the theorem on Cohomology and Base Change we conclude immediately that $p_*\mathscr{O}(V)$ is locally free over S of the same rank as V. The natural map between locally free sheaves $p^*p_*\mathscr{O}(V) \to \mathscr{O}(V)$ induces isomorphisms over each fiber and hence by the Nakayama Lemma it is indeed an isomorphism. The desired F' is simply the vector bundle associated to $p_*\mathscr{O}(V)$. □

Now apply the lemma to $V = \mathscr{O}_{\mathbb{P}_S(F)}(1) \otimes N_{Z/X}$, and we conclude that

$$N_{Z/X} \cong \mathscr{O}_{\mathbb{P}_S(F)}(-1) \otimes \bar{\psi}^*F'.$$

Therefore, on the blow-up $\phi : Y = \mathrm{Bl}_Z X \to X$,

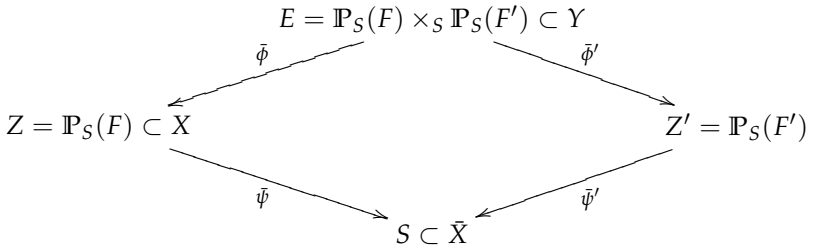
$$N_{E/Y} = \mathscr{O}_{\mathbb{P}_Z(N_{Z/X})}(-1).$$

[19] The reason that one can perform this blowing-down operation is an elementary calculation in the Mori theory, which I gleefully omit.

From the Euler sequence which defines the universal sub-line bundle we see easily that $\mathscr{O}_{\mathbb{P}_Z(L\otimes F)}(-1) = \bar{\phi}^*L \otimes \mathscr{O}_{\mathbb{P}_Z(F)}(-1)$ for any line bundle L over Z. Since the projectivization functor commutes with pull-backs, we have

$$E = \mathbb{P}_Z(N_{Z/X}) \cong \mathbb{P}_Z(\bar{\psi}^*F') = \bar{\psi}^*\mathbb{P}_S(F') = \mathbb{P}_S(F) \times_S \mathbb{P}_S(F').$$

For future reference we denote the projection map $Z' := \mathbb{P}_S(F') \to S$ by $\bar{\psi}'$ and $E \to Z'$ by $\bar{\phi}'$. The various sets and maps are summarized in the following commutative diagram.



with normal bundle of E in Y being

$$\begin{aligned} N_{E/Y} &= \mathscr{O}_{\mathbb{P}_Z(N_{Z/X})}(-1) = \mathscr{O}_{\mathbb{P}_Z(\mathscr{O}_Z(-1)\otimes\bar{\psi}^*F')}(-1) \\ &= \bar{\phi}^*\mathscr{O}_{\mathbb{P}_S(F)}(-1) \otimes \mathscr{O}_{\mathbb{P}_Z(\bar{\psi}^*F')}(-1) \\ &= \bar{\phi}^*\mathscr{O}_{\mathbb{P}_S(F)}(-1) \otimes \bar{\phi}'^*\mathscr{O}_{\mathbb{P}_S(F')}(-1). \end{aligned}$$

The upshot is that an ordinary $\mathbb{P}^r$ flop is locally, nearly the exceptional loci, determined by two rank $r+1$ vector bundles over a smooth variety S.

Remark 6.2. Notice that the bundles F and F' are uniquely determined up to a twisting by a line bundle. Namely, the pair (F, F') is equivalent to $(F \otimes L, F' \otimes L^*)$ for any line bundle L on S.

An ordinary flop is called *simple* if S is a point. In this case, $Z \cong Z' \cong \mathbb{P}^r$ and $E \cong \mathbb{P}^r \times \mathbb{P}^r$.

Theorem 6.3. *For an ordinary $\mathbb{P}^r$ flop $f : X \dashrightarrow X'$, the graph closure $\mathscr{F} := [\bar{\Gamma}_f]$ (induces a correspondence which) identifies the Chow motives. In particular, $\mathscr{F}$ preserves the Poincaré pairing on cohomology groups.*

It is important to me that CTC has to specify the identification of $H^*(X)$ and $H^*(X')$ in the very beginning. Furthermore, this identification only makes geometric sense if it comes from some sort of correspondence, like above, or of Fourier–Mukai type. With this in mind, we can state the CTC as follows.

Conjecture 6.4 (CTC for ordinary flops)**.** *GWT is invariant under ordinary flops, after an analytic continuation, determined by $\mathscr{F}$, over the Novikov variable q^ℓ corresponding to the rational curve contracted by ψ.*

The important thing here is that CTC is completely determined by $\mathscr{F}$. That includes, in particular, the analytic continuation.[20] Therefore, GWT of X' is uniquely determined by GWT of X, and vice versa, once we know $\mathscr{F}$. This is deterministic point of view. There is no variable to tweak, no parameter to unwind. Unlike the Standard Model in physics, there are at least 19 "free" parameters!

In this case X and X' both have one contracted curve ℓ and ℓ', and $\mathscr{F}(\ell) = -\ell'$. Therefore, the analytic continuation should take q^ℓ to $q^{-\ell'}$. However, the Novikov variable would only makes sense for effective curve classes: $q^{-\ell'}$ does not make sense naively. Therefore, one has to prove that the generating functions, when summing over all degrees, should be *analytic functions* in these variables. Otherwise, CTC wouldn't make sense. One may think that there is a "master function", analytic in $\in \mathbb{P}^1$, such that the GW generating functions of X and of X' are series expansions at $= 0$ and $= \infty$, with local expansion variables q^ℓ and $q^{\ell'}$ respectively.

Example 6.5.

$$\sum_{d=0} q^{d\ell} = \frac{1}{1-q^\ell}.$$

Now set q^ℓ to be $q^{-\ell'}$ on the RHS and re-expand at $q^{\ell'} = 0$:

$$\frac{1}{1-q^{-\ell'}} = \frac{q^{\ell'}}{q^{\ell'}-1} = -\sum_{d'=1} q^{d'\ell'}.$$

Theorem 6.6. *CTC for ordinary flops holds for simple flops in all genera, and genus zero in general.*

The analytic continuation is strictly necessary here. However, in some cases, the analytic continuation is "trivial", as in the case of Mukai flops, when the GWIs associated with the flopping curve vanish.

A contraction $(\psi, \bar{\psi}) : (X, Z) \to (\bar{X}, S)$ is of *Mukai type* if $Z = \mathbb{P}_S(F) \to S$ is a projective bundle under $\bar{\psi}$ and $N_{Z/X} = T^*_{Z/S}$. The corresponding algebraic flop $f : X \dashrightarrow X'$ exists and its local model can be realized as a *slice* of an ordinary flop. But, we will not pursue this...

Theorem 6.7. *Let $f : X \dashrightarrow X'$ be a Mukai flop. Then X and X' are diffeomorphic, and have isomorphic Hodge structures and full Gromov–Witten theory. In fact, any local Mukai flop is a limit of isomorphisms and all quantum corrections attached to the extremal ray vanish.*

These results are joint works with H.W. Lin and C.L. Wang.

6.3. **crepant resolution.** If X is smooth, and $\bar{X}$ has only *Gorenstein* quotient singularity, such that $\psi^* K_{\bar{X}} = K_X$, then we say that ψ is a *crepant resolution*.

[20] This is where we deviate most significantly from the formulation of Bryan–Graber. We ask an even stronger form than their formulation requires.

$\bar{X}$ has another "resolution" by an orbifold X'. That is, the "coarse moduli scheme" of X' is $\bar{X}$. The Gorenstein property means that the isotropy group at any point of X' acts trivially on $K_{X'}$.

First note that ψ here is very different from the flop case. While for flops, ψ is by definition small, here ψ must be divisorial (contracting divisors). This can be seen by the following simple argument. Since $\bar{X}$ has only quotient singularities, it must be $\mathbb{Q}$-factorial. Suppose ψ is small. Take an ample divisor A on X then $\psi^*\psi_*(A) = A$. On the other hand, any pullback divisor cannot be ample. We then have a contradiction, which means that ψ must be divisorial.

CTC is often named *crepant resolution conjecture* (CRC) in this special case. Originally, the idea that functoriality should happen in this case comes from physics. Y. Ruan was the first to formulate a mathematical statement of CRC, which was then refined and expanded by many others. The most important observation, by Coates, Corti, Iritani, Tseng, is that the *hard Lefschetz* property does not hold in general for orbifolds, and hence, one should not expect the *strong functoriality* to hold in general. So one has 2 choices: Either one can work in the subcategory the HL orbifolds, or one has to be content with *weak(er) functoriality*.

What is the hard Lefschetz property? Since $\bar{X}$ is projective, it has a very ample divisor ω. One can use ω *on* X' and ask whether the multiplicative operator by ω, denoted L_ω, satisfies the usual hard Lefschetz property:

$$L_\omega^k : H_{CR}^{(n-k)/2}(X') \xrightarrow{\cong} H_{CR}^{(n+k)/2}(X').$$

J. Fernandez proves that this is equivalent to the following condition.

Definition 6.8. *An orbifold X' satisfies the* hard Lefschetz *(HL) condition if the involution $\mathbf{I} : IX' \to IX'$ preserves the age.*

Let ℓ_i be the curves contracted by ψ. The strong version of CTC:

Conjecture 6.9. *Suppose that the orbifold X' satisfies the HL condition, and $\bar{X}$ admits a crepant resolution by X. There exists a graded linear isomorphism $\mathscr{F} : H^*(X) \to H^*_{CR}(X')$ and specific algorithm of specializing the Novikov variables $q^{\ell_i} = c_i$, such that the ancestor potential of X will be equal to that of X' after $\mathscr{F}$, and analytic continuations on q^{ℓ_i} before specializing them.*

Remark 6.10. Firstly, the generating function should be analytic in q^{ℓ_i}'s, in order for the specialization to make sense. (A priori, it is only a formal function in Novikov variables.) Then the specialization of q^{ℓ_i} should be consistent with the $\mathscr{F}$. For example, if $\mathscr{F}(\ell_i) = \eta_i$ is an "orbifold point class" of CR degree one, η_i can be represented as a root of unity to which q^{ℓ_i} should specialize. There are tricky issues of the (different) analytic continuations which we will not discuss, because I don't know much about them.

How can $\mathscr{F}$ be determined? As we learned in the scheme case, it should come from some kind of correspondence. In the case of orbifolds, the correspondence should look simpler in K-theory, so it is useful to take that

perspective. Why? For example, the K-theory of $[Y/G]$ is nothing but the G-equivariant K-theory on Y. The Riemann–Roch, on the other hand, will take $K([Y/G])$ to $H^*_{CR}([Y/G])$, as discovered by Tetsuro Kawasaki.[21] Therefore the inertia stack $I[Y/G]$ must be involved in the cohomology theory and an easy correspondence in K-theory can "descend" to a complicated-looking one in cohomology theory.

Remark 6.11. H. Iritani finds convincing evidences of this folklore belief. As expected, that is also a good way to determine the specialization. Unfortunately, I don't understand very well of his theories, but one can ask him during the conference week.

6.4. **HLK.** But there are other K-equivalences of the flopping type which might possess the strong form of CTC. For example, one can study the flops for orbifolds. That is, both X and X' are orbifolds. It would be too stringent to require both X and X' satisfy HL condition. Once one thinks about this, one thing should be immediately clear: One only needs the *neighborhoods of the exceptional loci to satisfy HL condition!* I will call this *HLK condition*. I started to talk about this notion since 2007[22] but it did not take hold until H. Iritani subsequently proposed an (independent) generalization of this notion. In any case, the HL condition is definitely too stringent for any comparison of orbifolds. HLK, or Iritani's generalization, I think, is the right notion.

Note that the orbifold flops do *not* really fit into (the narrowly defined) CRC as ψ are small. However, one can still talk about CTC. Two groups have been working on that, independently and along different directions: Bohui Chen and co. are one; Cadman, Jiang and myself is another.

6.5. **When HLK doesn't hold.** So far, we have been talking about the case when HLK holds. There are vast cases when HLK doesn't hold, but there are still interesting things one can say about the (weak) functoriality. This has been worked out by CCIT and Ruan in the following form [3] in the setting of crepant resolution. That is, X' is an orbifold and $\bar{X}$ its coarse moduli scheme. $\psi : X \to \bar{X}$ is a crepant resolution. Recall that ψ contracts divisors.

Conjecture 6.12. *There is a degree-preserving $\mathbb{C}[z, z^{-1}]]$-linear symplectic isomorphism $\mathscr{F} : \mathcal{H}_X \to \mathcal{H}_{X'}$ and a choice of analytic continuations of $\mathcal{L}_X$ and $\mathcal{L}_{X'}$ such that $\mathscr{F}(\mathcal{L}_X) = \mathcal{L}_{X'}$. Furthermore, $\mathscr{F}$ satisfies*

(1) $\mathscr{F}(\mathbf{1}_X) = \mathbf{1}_{X'}$.
(2) *$\mathscr{F}$ preserves the ring structure for the contracted divisors of ψ and ψ' (dual to ℓ and ell$'$).*
(3) $\mathscr{F}(\mathcal{H}_X^+) \oplus \mathcal{H}_{X'}^- = \mathcal{H}_{X'}$.
(4) *$\mathscr{F}(\phi_\mu)$ does not involve Novikov variables.*

[21] It is fair to say that Kawasaki was the first one to point out, in 1979, that the orbifold cohomology groups must be the cohomology of inertia stacks.

[22] For example in a KIAS conference in 2008.

Remark 6.13. The second condition does *not* "fit" into CTC for flops, as the flopping contractions are small.

6.6. **How to prove CTC (for ordinary flops)?** How to prove CTC in the HLK case? Well, one obvious way, of course, is to compute both sides and then, perhaps after tweaking a few parameters, identify them. I have nothing to say about it. What I am particularly interested in is when neither side can be computed explicitly, even in principle. An ordinary flop with an arbitrary base manifold S present such a challenge, so I will use it as a guiding example.

Now X and X' are birational. Naively, one may wish to "decompose" the varieties into the neighborhoods of exceptional loci and their complements. As the latter's are obviously isomorphic, one is reduced to study the local case. The degeneration formula provides a rigorous formulation of the above naive picture. Hence, it is enough to prove CTC for projective local models, which are related by the same type of crepant transformations. Of course, there is an issue about the relative invariants and absolute invariants. Those have to be worked out as well.

If the local models are computable, one might be able to compute both sides and check. In the case of ordinary flops, the local models are determined by a pair of arbitrary vector bundles F, F' of rank $r+1$ over an arbitrary base S. There is no hope of computing it explicitly as there is no explicit geometry even for the local models, which are double projective bundles. Nevertheless, one can take the following steps.

First, by suitable degeneration, one can reduce the statement to the special case when F and F' are split bundles.[23]

Then one can use localization theorem. Note that there is a torus T action on the double projective bundles. The fixed loci are sections of S. So the GWIs of the local models are in principle reducible to GWIS of S. This localization formulation has been worked out by J. Brown (and Givental).

APPENDIX A. QUANTIZATION AND HIGHER GENUS AXIOMATIC THEORY

A.1. **Preliminaries on quantization.** To quantize an infinitesimal symplectic transformation, or its corresponding quadratic hamiltonians, we recall the standard Weyl quantization. A polarization $\mathcal{H} = T^*\mathcal{H}_q$ on the symplectic vector space $\mathcal{H}$ (the phase space) defines a configuration space $\mathcal{H}_q$. The quantum "Fock space" will be a certain class of functions $f(\hbar, q)$ on $\mathcal{H}_q$ (containing at least polynomial functions), with additional formal variable $\hbar$ ("Planck's constant"). The classical observables are certain functions of p, q. The quantization process is to find for the classical mechanical system on $\mathcal{H}$ a "quantum mechanical" system on the Fock space such that the

[23] Actually, by blowing up and blowing down, one can reduce further to the case when (all line bundle factors in) F and F' are globally generated. However, this step isn't used in the proof.

classical observables, like the hamiltonians $h(q,p)$ on $\mathcal{H}$, are quantized to become operators $\widehat{h}(q, \frac{\partial}{\partial q})$ on the Fock space.

Let $A(z)$ be an $\mathrm{End}(H)$-valued Laurent formal series in z satisfying

$$(A(-z)f(-z), g(z)) + (f(-z), A(z)g(z)) = 0,$$

then $A(z)$ defines an infinitesimal symplectic transformation

$$\Omega(Af, g) + \Omega(f, Ag) = 0.$$

An infinitesimal symplectic transformation A of $\mathcal{H}$ corresponds to a quadratic polynomial[24] $P(A)$ in p, q

$$P(A)(f) := \frac{1}{2}\Omega(Af, f).$$

Choose a Darboux coordinate system $\{q_k^i, p_k^i\}$. The quantization $P \mapsto \widehat{P}$ assigns

$$\begin{aligned}
&\widehat{1} = 1,\ \widehat{p_k^i} = \sqrt{\hbar}\frac{\partial}{\partial q_k^i},\ \widehat{q_k^i} = q_k^i/\sqrt{\hbar},\\
&\widehat{p_k^i p_l^j} = \widehat{p_k^i}\,\widehat{p_l^j} = \hbar\frac{\partial}{\partial q_k^i}\frac{\partial}{\partial q_l^j},\\
&\widehat{p_k^i q_l^j} = q_l^j\frac{\partial}{\partial q_k^i},\\
&\widehat{q_k^i q_l^j} = q_k^i q_l^j/\hbar,
\end{aligned} \tag{A.1}$$

In summary, the quantization is the process

$$\begin{array}{ccccc}
A & \mapsto & P(A) & \mapsto & \widehat{P(A)}\\
\text{inf. sympl. transf.} & \mapsto & \text{quadr. hamilt.} & \mapsto & \text{operator on Fock sp..}
\end{array}$$

It can be readily checked that the first map is a Lie algebra isomorphism: The Lie bracket on the left is defined by $[A_1, A_2] = A_1A_2 - A_2A_1$ and the Lie bracket in the middle is defined by Poisson bracket

$$\{P_1(p,q), P_2(p,q)\} = \sum_{k,i}\frac{\partial P_1}{\partial p_k^i}\frac{\partial P_2}{\partial q_k^i} - \frac{\partial P_2}{\partial p_k^i}\frac{\partial P_1}{\partial q_k^i}.$$

The second map is not a Lie algebra homomorphism, but is very close to being one.

Lemma A.1.

$$[\widehat{P_1}, \widehat{P_2}] = \widehat{\{P_1, P_2\}} + \mathcal{C}(P_1, P_2),$$

where the cocycle $\mathcal{C}$, in orthonormal coordinates, vanishes except

$$\mathcal{C}(p_k^i p_l^j, q_k^i q_l^j) = -\mathcal{C}(q_k^i q_l^j, p_k^i p_l^j) = 1 + \delta^{ij}\delta_{kl}.$$

[24]Due to the nature of the infinite dimensional vector spaces involved, the "polynomials" here might have infinite many terms, but the degrees remain finite.

Example A.2. Let $\dim H = 1$ and $A(z)$ be multiplication by z^{-1}. It is easy to see that $A(z)$ is infinitesimally symplectic.

$$\begin{aligned} P(z^{-1}) &= -\frac{q_0^2}{2} - \sum_{m=0}^{\infty} q_{m+1}p_m \\ \widehat{P(z^{-1})} &= -\frac{q_0^2}{2} - \sum_{m=0}^{\infty} q_{m+1}\frac{\partial}{\partial q_m}. \end{aligned} \tag{A.2}$$

Note that one often has to quantize the symplectic instead of the infinitesimal symplectic transformations. Following the common practice in physics, define

$$\widehat{e^{A(z)}} := e^{\widehat{A(z)}}, \tag{A.3}$$

for $e^{A(z)}$ an element in the twisted loop group.

A.2. **τ-function for the axiomatic theory.** Let X be the space of N points and $H^{Npt} := H^*(X)$. Let T_i be the delta-function at the i-th point. Then $\{T_i\}_{i=1}^N$ form an orthonormal basis and are the idempotents of the quantum product

$$T_i * T_j = \delta_{ij} T_i.$$

The genus zero potential for N points is nothing but a sum of genus zero potentials of a point

$$F_0^{Npt}(t^1, \ldots, t^N) = F_0^{pt}(t^1) + \ldots + F_0^{pt}(t^N).$$

In particular, the genus zero theory of N points is semisimple.

By Theorem 3.6, any *semisimple* genus zero axiomatic theory T of rank N can be obtained from H^{Npt} by action of an element O^T in the twisted loop group. By Birkhoff factorization, $O^T = S^T(z^{-1})R^T(z)$, where $S(z^{-1})$ (resp. $R(z)$) is a matrix-valued function in z^{-1} (resp. z).

In order to define the axiomatic higher genus potentials G_g^T for the semisimple theory T, one first introduces the "τ-function of T".

Definition A.3. [5] *Define the* axiomatic τ-function *as*

$$\tau_G^T := \widehat{S^T}(\widehat{R^T}\tau_{GW}^{Npt}), \tag{A.4}$$

where τ_{GW}^{Npt} *is defined as*

$$\tau_{GW}^X := e^{\sum_{g=0}^{\infty}\hbar^{g-1}F_g^X}.$$

Define the axiomatic genus g potential G_g^T *via the formula*

$$\tau_G^T =: e^{\sum_{g=0}^{\infty}\hbar^{g-1}G_g^T}. \tag{A.5}$$

Remarks. (i) It is not obvious that the above definitions make sense. The function $\widehat{S^T}(\widehat{R^T}\tau^{Npt})$ is well-defined, due to some finiteness properties of τ^{pt}, called the $(3g-2)$-jet properties [5]. The fact that $\log \tau_G^T$ can be written

as $\sum_{g=0}^{\infty}\hbar^{g-1}(\text{formal function in } t)$ is also nontrivial. The interested readers are referred to the original article [5] or [10] for details.

(ii) What makes Givental's axiomatic theory especially attractive are the facts that

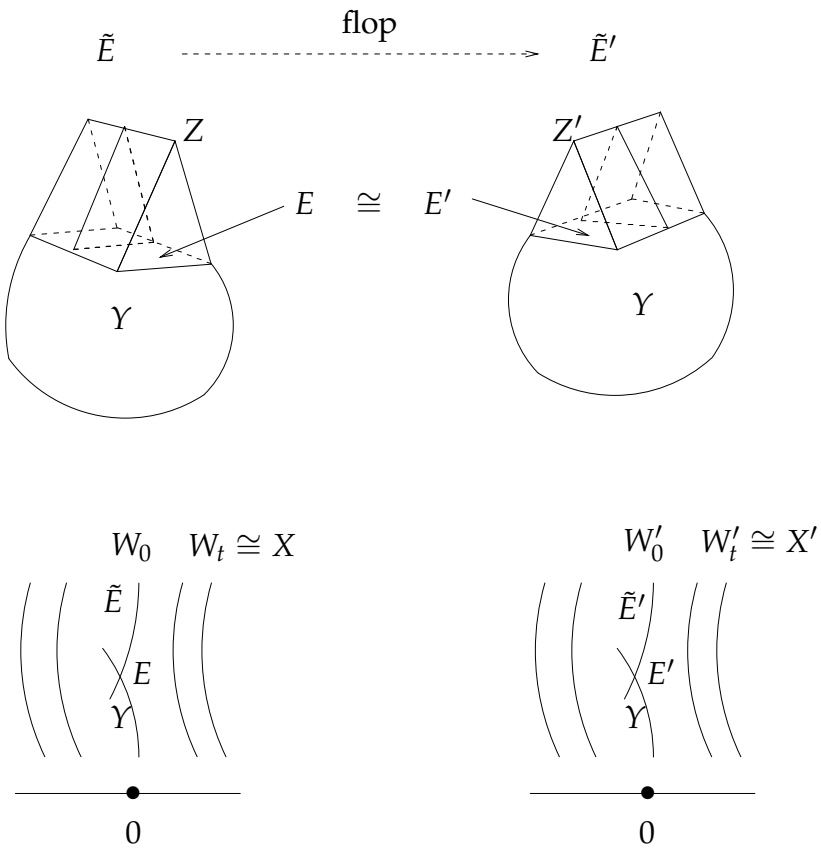
(a) It works for any semisimple Frobenius manifolds, not necessarily coming from geometry.
(b) It enjoys properties often complementary to the geometric theory.

Appendix B. Degeneration analysis for simple flops

As mentioned earlier, the local models for simple flops are double projective bundles over projective spaces, and are toric. So at least in principle, they are computable. In this section, we explain in details how to reduce a statement of CTC to the local models.

B.1. **The degeneration analysis.** Given an ordinary flop $f : X \dashrightarrow X'$, we apply degeneration to the normal cone to both X and X'. Then $Y_1 \cong Y_1' \cong Y$ and $E = E'$, by the definition of ordinary flops. The following notations will be used

$$Y := \mathrm{Bl}_Z X \cong Y_1 \cong Y_1', \quad \tilde{E} := \mathbb{P}_Z(N_{Z/X} \oplus \mathscr{O}), \quad \tilde{E}' := \mathbb{P}_{Z'}(N_{Z'/X'} \oplus \mathscr{O}).$$



Degeneration to the normal cone for ordinary flops.

Remark B.1. For simple $\mathbb{P}^r$ flops, $Y_2 \cong \mathbb{P}_{\mathbb{P}^r}(\mathscr{O}(-1)^{\oplus(r+1)} \oplus \mathscr{O}) \cong Y_2'$. However the gluing maps of Y_1 and Y_2 along E for X and X' differ by a twist which interchanges the order of factors in $E = \mathbb{P}^r \times \mathbb{P}^r$. Thus $W_0 \ncong W_0'$ and it is necessary to study the details of the degenerations. In general, f induces an ordinary flop $\tilde{f} : Y_2 \dashrightarrow Y_2'$ of the same type which is the local model of f.

B.2. **Liftings of cohomology insertions.** Next we discuss the presentation of $\alpha(0)$. Denote by $\iota_1 \equiv j : E \hookrightarrow Y_1 = Y$ and $\iota_2 : E \hookrightarrow Y_2 = \tilde{E}$ the natural inclusions. The class $\alpha(0)$ can be represented by $(j_1^*\alpha(0), j_2^*\alpha(0)) = (\alpha_1, \alpha_2)$ with $\alpha_i \in H^*(Y_i)$ such that

$$\iota_1^*\alpha_1 = \iota_2^*\alpha_2 \quad \text{and} \quad \phi_*\alpha_1 + p_*\alpha_2 = \alpha. \tag{B.1}$$

Such representatives are called *liftings* which are by no means unique. The flexibility on different choices will be useful.

One choice of the lifting is

$$\alpha_1 = \phi^*\alpha \quad \text{and} \quad \alpha_2 = p^*(\alpha|_Z), \tag{B.2}$$

since they satisfy the conditions (B.1): (α_1, α_2) restrict to the same class in E and push forward to α and 0 in X respectively. More generally:

Lemma B.2. *Let $\alpha(0) = (\alpha_1, \alpha_2)$ be a choice of lifting. Then*

$$\alpha(0) = (\alpha_1 - \iota_{1*}e, \alpha_2 + \iota_{2*}e)$$

is also a lifting for any class e in E of the same dimension as α. Moreover, any two liftings are related in this manner. In particular, α_1 and α_2 are uniquely determined by each other.

Proof. The first statement follows from the facts that

$$\iota_1^*\iota_{1*}e = (e.c_1(N_{E/Y}))_E = -(e.c_1(N_{E/\tilde{E}}))_E = -\iota_2^*\iota_{2*}e$$

and $-\phi_*\iota_{1*}e + p_*\iota_{2*}e = 0$ (since $\phi \circ \iota_1 = p \circ \iota_2 = \bar{\phi} : E \to Z$).

For the second statement, let (α_1, α_2) and (a_1, a_2) be two liftings. From

$$\phi_*(\alpha_1 - a_1) = -p_*(\alpha_2 - a_2) \in H^*(Z),$$

we have that $\phi^*\phi_*(\alpha_1 - a_1)$ is a class in E. Hence $\alpha_1 - a_1 = \iota_{1*}e$ for $e \in H^*(E)$. It remains to show that if (a_1, a_2) and $(a_1, \tilde{a}_2)$ are two liftings then $a_2 = \tilde{a}_2$. Indeed by (B.1), $\iota_2^*(a_2 - \tilde{a}_2) = 0$. Hence by Lemma B.3 below $z := a_2 - \tilde{a}_2 \in i_*H^*(Z)$. By (B.1) again $z = p_*z = p_*(a_2 - \tilde{a}_2) = 0$. □

For an ordinary flop $f : X \dashrightarrow X'$, we compare the degeneration expressions of X and X'. For a given admissible triple $\eta = (\Gamma_1, \Gamma_2, I_\rho)$ on the degeneration of X, one may pick the corresponding $\eta' = (\Gamma_1', \Gamma_2', I_\rho')$ on the degeneration of X' such that $\Gamma_1 = \Gamma_1'$. Since

$$\phi^*\alpha - \phi'^*\mathscr{F}\alpha \in \iota_{1*}H^*(E) \subset H^*(Y),$$

Lemma B.2 implies that one can choose $\alpha_1 = \alpha_1'$. This can be done, for example, by modifying the choice of (B.2) $j_1^*\alpha(0) = \phi^*\alpha$ and $j_1'^*\mathscr{F}\alpha(0) = \phi'^*\mathscr{F}\alpha$ by adding suitable classes in E to make them equal. The above procedures identify relative invariants on the $Y_1 = Y = Y_1'$ from both sides term by term, and we are left with the comparison of the corresponding relative invariants on $\tilde{E}$ and $\tilde{E}'$. The following simple lemma is useful.

Lemma B.3. *Let $\tilde{E} = \mathbb{P}_Z(N \oplus \mathscr{O})$ be a projective bundle with base $i : Z \hookrightarrow \tilde{E}$ and infinity divisor $\iota_2 : E = \mathbb{P}_Z(N) \hookrightarrow \tilde{E}$. Then the kernel of the restriction map $\iota_2^* : H^*(\tilde{E}) \to H^*(E)$ is $i_*H^*(Z)$.*

Proof. $i_*H^*(Z)$ obviously lies in the kernel of ι_2^*. The fact it is the entire kernel can be seen, for example, by a dimension count. □

The ordinary flop f induces an ordinary flop

$$\tilde{f} : \tilde{E} \dashrightarrow \tilde{E}'$$

on the local model. Moreover $\tilde{f}$ may be considered as a family of simple ordinary flops $\tilde{f}_t : \tilde{E}_t \dashrightarrow \tilde{E}_t'$ over the base S, where $t \in S$ and $\tilde{E}_t$ is the fiber of $\tilde{E} \to Z \to S$ etc.. Denote again by $\mathscr{F}$ the cohomology correspondence induced by the graph closure. Then

Proposition B.4 (Cohomology reduction to local models)**.** *Let $f : X \dashrightarrow X'$ be a $\mathbb{P}^r$ flop over base S with $\dim S = s$. Let $\alpha \in H^*(X)$ with liftings $\alpha(0) = (\alpha_1, \alpha_2)$ and $\mathscr{F}\alpha(0) = (\alpha_1', \alpha_2')$. Then*

$$\alpha_1 = \alpha_1' \quad \Longleftrightarrow \quad \mathscr{F}\alpha_2 = \alpha_2'.$$

Proof. Let $\alpha \in H^{2l}(X)$ with $l \in \frac{1}{2}\mathbb{N}$. If $l > \dim Z = r + s$ then $\alpha|_Z = 0$. By (B.2) and Lemma B.2, all liftings take the form $\alpha(0) = (\alpha - \iota_{1*}e, \iota_{2*}e)$ and $\mathscr{F}\alpha(0) = (\alpha - \iota_{1*}'e', \iota_{2*}'e')$ for e, e' being classes in E. In this case the proof is trivial since $\mathscr{F}$ is the identity map on $H^*(E)$. So we may assume that $l \le r + s$.

($\Rightarrow$) From the contact order condition $\iota_2^*\alpha_2 = \iota_1^*\alpha_1 = \iota_1'^*\alpha_1' = \iota_2'^*\alpha_2'$ and the fact that $\tilde{f}$ is an isomorphism outside Z, we get

$$\iota_2'^*(\mathscr{F}\alpha_2 - \alpha_2') = \mathscr{F}\iota_2^*\alpha_2 - \iota_2'^*\alpha_2' = \iota_2^*\alpha_2 - \iota_2'^*\alpha_2' = 0.$$

Thus $\mathscr{F}\alpha_2 - \alpha_2' = i_*'z'$ for some $z' \in H^{2(l-(r+1))}(Z')$ (where $i' : Z' \hookrightarrow \tilde{E}'$) by Lemma B.3 and the fact that $\operatorname{codim}_{\tilde{E}'} Z' = r + 1$.

For simple flops, $s = 0$ and then $l - (r+1) \le s - 1 < 0$. So $z' = 0$ and we are done. In general we restrict the equation to each fiber $\tilde{f}_t : \tilde{E}_t \to \tilde{E}_t'$. Since $\bar{\Gamma}_{\tilde{f}}|_t = \bar{\Gamma}_{\tilde{f}_t}$, by the case of simple flops we get $(\mathscr{F}\alpha_2 - \alpha_2')|_{\tilde{E}_t'} = 0$ for all $t \in S$. That is, z' is a class supported in the fiber of $p' : Z' \to S$. But then $\operatorname{codim}_{\tilde{E}'} z' \ge s + r + 1 > l$, which implies that $z' = 0$.

($\Leftarrow$) For ease of notations we omit the embedding maps of E into Y, $\tilde{E}$ and $\tilde{E}'$. By (B.2) and Lemma B.2 we have $\alpha_1 = \phi^*\alpha - e_1$ and $\alpha_1' = \phi'^*\mathscr{F}\alpha - e_1'$ for some classes e_1, e_1' in E. Thus $\alpha_1' = \alpha_1 - e$ for some class e in E. By Lemma B.2 again $\alpha(0)$ has a lifting $(\alpha_1 - e, \alpha_2 + e) = (\alpha_1', \alpha_2 + e)$ and by the first part of this proposition we must have $\mathscr{F}(\alpha_2 + e) = \alpha_2'$. By assumption $\mathscr{F}\alpha_2 = \alpha_2'$, hence $\mathscr{F}e = 0$ and then $e = 0$. □

Remark B.5. Proposition B.4 (with cohomology groups replaced by Chow groups) leads to a proof of equivalence of Chow motives under ordinary flops. One has to establish the equivalence of Chow groups for simple flops, which is not too difficult. The degeneration to the normal cone then allows us to reduce the general case to the local case and then to the local simple case.

B.3. **Reduction to relative local models.** First notice that $A_1(\tilde{E}) = \iota_{2*}A_1(E)$ since both are projective bundles over Z. We then have

$$\phi^*\beta = \beta_1 + \beta_2$$

by regarding β_2 as a class in $E \subset Y$. Indeed $\phi_*(\beta_1 + \beta_2) = \phi_*\beta_1 + p_*\beta_2 = \beta$ and

$$((\beta_1 + \beta_2).E)_Y = (\beta_1.E)_Y - (\beta_2.E)_{\tilde{E}} = |\mu| - |\mu| = 0$$

(where $N_{E/\tilde{E}} \cong N_{E/Y}^*$ is used). These characterize the class $\phi^*\beta$.

We consider only the case $g = 0$. Define the generating series

$$\langle A \mid \varepsilon, \mu\rangle^{(\tilde{E},E)} := \sum_{\beta_2 \in NE(\tilde{E})} \frac{1}{|\mathrm{Aut}\,\mu|} \langle A \mid \varepsilon, \mu\rangle_{\beta_2}^{(\tilde{E},E)} q^{\beta_2}.$$

and the similar one with possibly disconnected domain curves

$$\langle A \mid \varepsilon, \mu\rangle^{\bullet(\tilde{E},E)} := \sum_{\Gamma;\, \mu_\Gamma = \mu} \frac{1}{|\mathrm{Aut}\,\Gamma|} \langle A \mid \varepsilon, \mu\rangle_{\Gamma}^{\bullet(\tilde{E},E)} q^{\beta^\Gamma}.$$

Proposition B.6. *To prove $\mathscr{F}\langle\alpha\rangle^X \cong \langle\mathscr{F}\alpha\rangle^{X'}$ (for all α), it is enough to show that*

$$\mathscr{F}\langle A \mid \varepsilon, \mu\rangle^{(\tilde{E},E)} \cong \langle \mathscr{F} A \mid \varepsilon, \mu\rangle^{(\tilde{E}',E)} \tag{B.3}$$

for all A, ε, μ.

Proof. For the n-point function $\langle\alpha\rangle^X = \sum_{\beta\in NE(X)} \langle\alpha\rangle_\beta^X q^\beta$, the degeneration formula gives

$$\begin{aligned}\langle\alpha\rangle^X &= \sum_{\beta\in NE(X)} \sum_{\eta\in\Omega_\beta} \sum_I C_\eta \langle\alpha_1 \mid e_I, \mu\rangle_{\Gamma_1}^{\bullet(Y_1,E)} \langle\alpha_2 \mid e^I, \mu\rangle_{\Gamma_2}^{\bullet(Y_2,E)} q^{\phi^*\beta} \\ &= \sum_\mu \sum_I \sum_{\eta\in\Omega_\mu} C_\eta \left(\langle\alpha_1 \mid e_I, \mu\rangle_{\Gamma_1}^{\bullet(Y_1,E)} q^{\beta_1}\right) \left(\langle\alpha_2 \mid e^I, \mu\rangle_{\Gamma_2}^{\bullet(Y_2,E)} q^{\beta_2}\right).\end{aligned}$$

To simplify the generating series, we consider also absolute invariants $\langle\alpha\rangle^{\bullet X}$ with possibly disconnected domain curves as before. Then by comparing the order of automorphisms,

$$\langle\alpha\rangle^{\bullet X} = \sum_\mu m(\mu) \sum_I \langle\alpha_1 \mid e_I, \mu\rangle^{\bullet(Y_1,E)} \langle\alpha_2 \mid e^I, \mu\rangle^{\bullet(Y_2,E)}.$$

To compare $\mathscr{F}\langle\alpha\rangle^{\bullet X}$ and $\langle\mathscr{F}\alpha\rangle^{\bullet X'}$, by Proposition B.4 we may assume that $\alpha_1 = \alpha_1'$ and $\alpha_2' = \mathscr{F}\alpha_2$. This choice of cohomology liftings identifies the relative invariants of (Y_1, E) and those of (Y_1', E') with the same topological types. It remains to compare

$$\langle\alpha_2 \mid e^I, \mu\rangle^{\bullet(\tilde{E},E)} \quad \text{and} \quad \langle\mathscr{F}\alpha_2 \mid e^I, \mu\rangle^{\bullet(\tilde{E}',E)}.$$

We further split the sum into connected invariants. Let Γ^π be a connected part with the contact order μ^π induced from μ. Denote $P : \mu = \sum_{\pi\in P} \mu^\pi$ a partition of μ and $P(\mu)$ the set of all such partitions. Then

$$\langle A \mid \varepsilon, \mu\rangle^{\bullet(\tilde{E},E)} = \sum_{P\in P(\mu)} \prod_{\pi\in P} \sum_{\Gamma^\pi} \frac{1}{|\mathrm{Aut}\,\mu^\pi|} \langle A^\pi \mid \varepsilon^\pi, \mu^\pi\rangle_{\Gamma^\pi}^{(\tilde{E},E)} q^{\beta^{\Gamma^\pi}}.$$

If one fixes the above data in the summation of (B.3), then the only index to be summed over is β^{Γ^π} on $\tilde{E}$. This reduces the problem to $\langle A^\pi \mid \varepsilon^\pi, \mu^\pi\rangle^{(\tilde{E},E)}$. □

Remark B.7. Here is a brief comment on the term

$$\mathscr{F}\langle\alpha_2 \mid e^I, \mu\rangle^{(\tilde{E},E)} = \sum_{\beta_2 \in NE(\tilde{E})} \frac{1}{|\mathrm{Aut}\,\mu|} \langle\alpha_2 \mid e^I, \mu\rangle_{\beta_2}^{(\tilde{E},E)} q^{\mathscr{F}\beta_2}.$$

Since $\tilde{E}$ is a projective bundle, $NE(\tilde{E}) = i_* NE(Z) \oplus \mathbb{Z}_+\gamma$ with γ the fiber line class of $\tilde{E} \to Z$. The point is that, for $\beta_2 \in NE(\tilde{E})$ it is in general not true that $\mathscr{F}\beta_2 \equiv \beta_2$ (in E) is effective in $\tilde{E}'$.

Indeed, for simple ordinary flops, let $\gamma = \delta'$, $\delta = \gamma'$ be the two line classes in $E \cong \mathbb{P}^r \times \mathbb{P}^r$. It is easily checked that $\ell \sim \delta - \gamma$ in $\tilde{E}$. Hence $\ell = -\ell'$ and $\gamma = \gamma' + \ell'$ and

$$\beta_2 = d_1\ell + d_2\gamma = (d_2 - d_1)\ell' + d_2\gamma'.$$

$\mathscr{F}\beta_2 \in NE(\tilde{E}')$ if and only if $d_2 \geq d_1$. Therefore,

$$\langle\alpha_2 \mid e^I, \mu\rangle^{(\tilde{E},E)} = \langle\mathscr{F}\alpha_2 \mid e^I, \mu\rangle^{(\tilde{E}',E)}$$

cannot possibly hold term by term. Analytic continuations are in general needed.

B.4. **Relative to absolute.** Recall that we are now in the local relative case, with $X = \tilde{E}$. We shall combine a method of Maulik and Pandharipande to further reduce the relative cases to the absolute cases with at most descendent insertions along E. Following them, we call the pair

$$(\varepsilon, \mu) = \{(\varepsilon_1, \mu_1), \cdots, (\varepsilon_\rho, \mu_\rho)\}$$

with $\varepsilon_i \in H^*(E)$, $\mu_i \in \mathbb{N}$ a *weighted partition*, a partition of contact orders weighted by cohomology classes in E.

Proposition B.8. *For an ordinary flop $\tilde{E}$ ⇢ $\tilde{E}'$, to prove*

$$\mathscr{F}\langle A \mid \varepsilon, \mu\rangle \cong \langle\mathscr{F}A \mid \varepsilon, \mu\rangle$$

for any A and (ε, μ), it is enough to show that

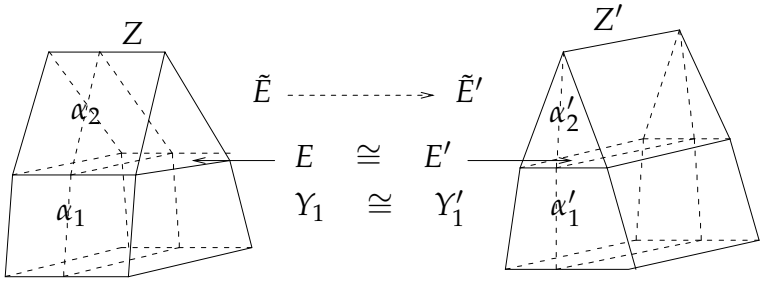
$$\mathscr{F}\langle A, \tau_{k_1}\varepsilon_1, \ldots, \tau_{k_\rho}\varepsilon_\rho\rangle^{\tilde{E}} \cong \langle\mathscr{F}A, \tau_{k_1}\varepsilon_1, \ldots, \tau_{k_\rho}\varepsilon_\rho\rangle^{\tilde{E}'}$$

for any possible insertions $A \in H^*(\tilde{E})^{\oplus n}$, $k_j \in \mathbb{N} \cup \{0\}$ and $\varepsilon_j \in H^*(E)$. (Here we abuse the notations and denote $\iota_{2*}\,\varepsilon \in H^*(\tilde{E})$ by the same symbol ε.)

The rest of this subsection is devoted to the proof of this proposition which proceeds inductively on the triple $(|\mu|, n, \rho)$ in the lexicographical order with ρ in the reverse order. Given $\langle\alpha_1, \ldots, \alpha_n \mid \varepsilon, \mu\rangle$, since $\rho \leq |\mu|$, it is clear that there are only finitely many triples of lower order. The proposition holds for those cases by the induction hypothesis.

We apply degeneration to the normal cone for $Z \hookrightarrow \tilde{E}$ to get $W \to \mathbb{A}^1$. Then $W_0 = Y_1 \cup Y_2$ with $\pi : Y_1 \cong \mathbb{P}_E(\mathscr{O}_E(-1,-1) \oplus \mathscr{O}) \to E$ a $\mathbb{P}^1$ bundle and $Y_2 \cong \tilde{E}$. Denote by $E_0 = E = Y_1 \cap Y_2$ and $E_\infty \cong E$ the zero and infinity divisors of Y_1 respectively. The idea is to analyze the degeneration formula for $\langle\alpha_1, \ldots, \alpha_n, \tau_{\mu_1 - 1}\varepsilon_1, \ldots, \tau_{\mu_\rho - 1}\varepsilon_\rho\rangle^{\tilde{E}}$. We follow the procedure used in the proof of Proposition B.6 to split the generating series of invariants with

possibly disconnected domain curves, according to the contact order. For $\beta = d_1\ell + d_2\gamma \in NE(\tilde{E})$, $c_1(\tilde{E}).\beta = d_2 c_1(\tilde{E}).\gamma$, hence by the virtual dimension counting d_2 is uniquely determined for a given generating series with fixed cohomology insertions.



Degeneration to normal cone for local models.

We observe that during the splitting of β's, the "main terms" with the highest total contact order only occur when the curve classes in Y_1 are fiber classes. Indeed, let (β_1, β_2) be a splitting of β. Since

$$NE(Y_1) = \mathbb{Z}_+\delta + \mathbb{Z}_+\bar{\gamma} + \mathbb{Z}_+\gamma \quad \text{and} \quad NE(Y_2) = \mathbb{Z}_+\ell + \mathbb{Z}_+\gamma$$

($\bar{\gamma}$ is the fiber class of Y_1), we have

$$(\beta_1, \beta_2) = (a\delta + b\gamma + c\bar{\gamma}, d\ell + e\gamma)$$

subject to

$$a, b, c, d, e \geq 0, \quad a + d = d_1, \quad c = d_2$$

and the total contact order condition

$$e = (\beta_2.E)_{\tilde{E}} = (\beta_1.E)_{Y_1} = -a - b + c.$$

In particular, $e \leq d_2$ with $e = d_2$ if and only if that $a = b = 0$. In this case $\beta_1 = d_2\bar{\gamma}$ and the invariants on (Y_1, E) are fiber class integrals.

It is sufficient to consider $(\varepsilon_1, \ldots, \varepsilon_\rho) = e_I = (e_{i_1}, \ldots, e_{i_\rho})$. Since $\varepsilon_i|_Z = 0$, one may choose the cohomology lifting $\varepsilon_i(0) = (\iota_{1*}\,\varepsilon_i, 0)$. This ensures that insertions of the form $\tau_k\,\varepsilon$ must go to the Y_1 side in the degeneration formula.

Lemma B.9. *For a general cohomology insertion $\alpha \in H^*(\tilde{E})$, the lifting can be chosen to be $\alpha(0) = (a, \alpha)$ for some a.*

Proof. $\alpha(0)$ may be chosen as $(\phi^*\alpha, p^*(\alpha|_Z))$. Since $(\alpha - p^*(\alpha|_Z)).Z = 0$, the class $e := \alpha - p^*(\alpha|_Z)$ can be taken to be supported in E. Then Lemma B.2 implies that $\alpha(0)$ can be modified to be $(\phi^*\alpha - e, \alpha)$. □

¿From $\alpha(0) = (a, \alpha)$ and $\mathscr{F}\alpha(0) = (a', \mathscr{F}\alpha)$, Lemma B.3 implies that $a = a'$. As before the relative invariants on (Y_1, E) can be regarded as constants

under $\mathscr{F}$. Then

$$\langle \alpha_1, \dots, \alpha_n, \tau_{\mu_1-1} e_{i_1}, \dots, \tau_{\mu_\rho-1} e_{i_\rho} \rangle^{\bullet \tilde{E}} = \sum_{\mu'} m(\mu') \times$$

$$\sum_{I'} \langle \tau_{\mu_1-1} e_{i_1}, \dots, \tau_{\mu_\rho-1} e_{i_\rho} \mid e^{I'}, \mu' \rangle^{\bullet (Y_1, E)} \langle \alpha_1, \dots, \alpha_n \mid e_{I'}, \mu' \rangle^{(\tilde{E}, E)} + R,$$

where R denotes the remaining terms which either have total contact order smaller than d_2 or have number of insertions fewer than n on the $(\tilde{E}, E)$ side or the invariants on $(\tilde{E}, E)$ are disconnected ones.

For the main terms, we claim that the total contact order $d_2 = |\mu'|$ equals $|\mu| = \sum_{i=1}^{\rho} \mu_i$. This follows from the dimension counting on $\tilde{E}$ and $(\tilde{E}, E)$. Indeed let $D = c_1(\tilde{E}).\beta + \dim \tilde{E} - 3$. For the absolute invariant on $\tilde{E}$,

$$\sum_{j=1}^{n} \deg \alpha_j + |\mu| - \rho + \sum_{j=1}^{\rho} (\deg e_{i_j} + 1) = D + n + \rho,$$

while on $(\tilde{E}, E)$ (notice that now $c_1(\tilde{E}).\beta_2 = d_2 c_1(\tilde{E}).\gamma = c_1(\tilde{E}).\beta$),

$$\sum_{j=1}^{n} \deg \alpha_j + \sum_{j=1}^{\rho'} \deg e_{i'_j} = D + n + \rho' - |\mu'|.$$

Hence (e_I, μ) appears as one of the $(e_{I'}, \mu')$'s and $|\mu| = |\mu'| = d_2$.

In particular, R is $\mathscr{F}$-invariant by induction. Moreover,

$$\deg e_I - \deg e_{I'} = \rho - \rho'.$$

We will show that the highest order term in the sum consists of the single term

$$C(\mu) \langle \alpha_1, \dots, \alpha_n \mid e_I, \mu \rangle^{(\tilde{E}, E)}$$

where $C(\mu) \neq 0$.

For any $(e_{I'}, \mu')$ in the highest order term, consider the splitting of weighted partitions

$$(e_I, \mu) = \coprod_{k=1}^{\rho'} (e_{I^k}, \mu^k)$$

according to the connected components of the relative moduli of (Y_1, E), which are indexed by the contact points of μ' by the genus zero assumption and the fact that the invariants on $(\tilde{E}, E)$ are connected invariants.

Since fiber class invariants on $\mathbb{P}^1$ bundles can be computed by pairing cohomology classes in E with GW invariants in the fiber $\mathbb{P}^1$, we must have $\deg e_{I^k} + \deg e^{i'_k} \leq \dim E$ to get non-trivial invariants. That is

$$\deg e_{I^k} = \sum_j \deg e_{i^k_j} \leq \dim E - \deg e^{i'_k} \equiv \deg e_{i'_k}$$

for each k. In particular, $\deg e_I \leq \deg e_{I'}$, hence also $\rho \leq \rho'$.

The case $\rho < \rho'$ is handled by the induction hypothesis, so we assume that $\rho = \rho'$ and then $\deg e_{I^k} = \deg e_{i'_k}$ for each $k = 1, \dots, \rho'$. In particular $I^k \neq \emptyset$ for each k. This implies that I^k consists of a single element. By reordering we may assume that $I^k = \{i_k\}$ and $(e_{I^k}, \mu^k) = \{(e_{i_k}, \mu_k)\}$.

Since the relative invariants on Y_1 are fiber integrals, the virtual dimension for each k (connected component of the relative virtual moduli) is

$$\begin{aligned} &2\mu_k' + \dim Y_1 - 3 + 1 + (1 - \mu_k') \\ &\quad = (\mu_k - 1) + (\deg e_{i_k} + 1) + (\dim E - \deg e_{i_k'}). \end{aligned}$$

Together with $\deg e_{i_k} = \deg e_{i_k'}$ this implies that

$$\mu_k' = \mu_k, \quad k = 1, \dots, \rho.$$

¿From the fiber class invariants consideration and

$$\deg e_{i_k} + \deg e^{i_k'} = \dim E,$$

e_{i_k} and $e^{i_k'}$ must be Poincaré dual to get non-trivial integral over E. That is, $e_{i_k'} = e_{i_k}$ for all k and $(e_{I'}, \mu') = (e_I, \mu)$. This gives the term we expect for with $C(\mu)$ a nontrivial fiber class invariant. The proof of Proposition B.8 is complete.

The functional equations for these special absolute invariants with descendents will be handled in §5.

B.5. **Examples.** We consider simple $\mathbb{P}^r$ flops for $r \leq 2$ in general and for $r \geq 3$ under nefness constraint on K_X.

If $\beta = d\ell$, the invariant depends only on Z, $\alpha|_Z$ and $N_{Z/X}$. In particular

$$\langle \alpha \rangle^X_{g,n,d\ell} = \langle p^*(\alpha|_Z) \rangle^{\tilde{E}}_{g,n,d\ell}.$$

Thus we consider $\beta \neq d\ell$. Let $\alpha_i \in H^{2l_i}(X)$. By the divisor axiom, we may assume that $l_i \geq 2$ for all i.

For $\eta = (\Gamma_1, \Gamma_2, I_\rho)$ associated to (g, n, β), let d, d_{Γ_1} and d_{Γ_2} be the virtual dimension (without marked points) of stable morphisms into X and relative stable morphisms into (Y_1, E), (Y_2, E) respectively. We have $l_1 + \cdots + l_n = d + n$. Moreover, since $\dim E = 2r$, the degeneration formula implies that $d = d_{\Gamma_1} + d_{\Gamma_2} - 2r\rho$.

We assume that the summand given by η is not zero. Since $\beta \neq d\ell$ and $A_1(Y_2)$ is spanned by ℓ and a fiber line γ, we see that $\beta_1 \neq 0$ and $\Gamma_1 \neq \emptyset$.

If $\rho = 0$ then $\Gamma_2 = \emptyset$ by connectedness, and this gives the blow-up term

$$\langle \tilde{\alpha} \rangle^Y_{g,n,\phi^*\beta}.$$

So we assume that $\rho \neq 0$. By reordering, we may assume that in the degeneration expression α_i appears in the Y_1 part for $1 \leq i \leq m$ and α_i appears in the Y_2 part for $m + 1 \leq i \leq n$. By transversality, the corresponding relative invariant is non-trivial only if $2 \leq l_i \leq r$ for $m + 1 \leq i \leq n$. If $r = 1$ this simply means that all α_i's appear in Y_1. In the following we abuse the notation by writing $|\mu|$ as μ.

Theorem B.10 (A. Li-Ruan)**.** *For simple $\mathbb{P}^1$ flops of threefolds with $\beta \neq d\ell$,*

$$\langle \alpha \rangle^X_{g,n,\beta} = \langle \tilde{\alpha} \rangle^Y_{g,n,\phi^*\beta} = \langle \mathscr{F}\alpha \rangle^{X'}_{g,n,\mathscr{F}\beta}.$$

That is, there are no degenerate terms and hence no analytic continuations are needed for non-exceptional curve classes.

Proof. If $r = 1$, then $(K_X.p_*\beta_2) = 0$, $d = -(K_X.\beta)$ and

$$\begin{aligned}(K_Y.\beta_1) &= (\phi^*K_X.\beta_1) + (E.\beta_1) = (K_X.\phi_*\beta_1) + \mu \\ &= (K_X.(\beta - p_*\beta_2)) + \mu = (K_X.\beta) + \mu.\end{aligned}$$

So

$$d_{\Gamma_1} = -(K_Y.\beta_1) + \rho - \mu = d + (\rho - 2\mu).$$

If $\rho \neq 0$ then $d_{\Gamma_1} < d$. Since $l_i \geq 2$, we may assume that α_i's are disjoint from Z, hence they must all contribute to the Y_1 part. This forces that $\rho = 0$ and the result follows. □

For simple $\mathbb{P}^2$ flops, non-trivial degenerate terms do occur even for $n \leq 3$ and $g = 0$. Let $v_i := |\Gamma_i|$ be the number of connected components.

Lemma B.11. *For $\tilde{E} = \mathbb{P}_Z(N \oplus \mathscr{O})$ of a pair $Z \subset X$,*

$$c_1(\tilde{E}) = (\operatorname{rk} N + 1)E + p^*c_1(X)|_Z.$$

Proof. Indeed, from $0 \to \mathscr{O} \to \mathscr{O}(1) \otimes p^*(N \oplus \mathscr{O}) \to T_{\tilde{E}/Z} \to 0$ we get $c_1(T_{\tilde{E}/Z}) = (\operatorname{rk} N + 1)E + p^*c_1(N)$, so the formula follows from $c_1(\tilde{E}) = c_1(T_{\tilde{E}/Z}) + p^*c_1(Z)$. □

Proposition B.12. *For simple $\mathbb{P}^2$ flops, let $n \leq 3$ and $\alpha_i \in H^{2l_i}(X)$ with $l_i \geq 2$ for $i = 1, \ldots, n$. Consider $\beta \neq d\ell$ and an admissible triple η with $\rho \neq 0$. Then $v_1 = \rho = \mu$, $v_2 = 1$ and $l_i = 2$ for all i.*

Proof. Since $c_1(Y_2) = 4E$ (by Lemma B.11), we find that

$$\begin{aligned}d_{\Gamma_2} &= 4(E.\beta_2) + 2v_2 + \rho - \mu \\ &= 3\mu + \rho + 2v_2.\end{aligned}$$

So $d_{\Gamma_2} - 4\rho = 3(\mu - \rho) + 2v_2 \geq 2$.

For one-point invariants, $l_1 = d + 1 = d_{\Gamma_1} + 3(\mu - \rho) + 2v_2 + 1 \geq d_{\Gamma_1} + 3$. It forces that α_1 contributes in Y_2, hence $l_1 = 2$ and $d = 1$. But $d_{\Gamma_1} \geq 0$ implies that $d \geq 2$, hence a contradiction.

For two-point invariants, from $l_1 + l_2 = d + 2 = d_{\Gamma_1} + 3(\mu - \rho) + 2v_2 + 2 \geq d_{\Gamma_1} + 4$ and the fact that α_i contributes to the Y_2 part in the degeneration formula only if $l_i = 2$, similar argument shows that the only non-trivial case is that $l_1 = l_2 = 2$ and both α_1 and α_2 contribute in Y_2. Moreover the equality holds hence that $\mu = \rho$, $v_2 = 1$ and $d_{\Gamma_1} = 0$.

We now consider three-point invariants. From

$$l_1 + l_2 + l_3 = d + 3 = d_{\Gamma_1} + (d_{\Gamma_2} - 4\rho) + 3 \geq d_{\Gamma_1} + 5,$$

if only α_3 contributes to Y_2 then $l_1 + l_2 \geq d_{\Gamma_1} + 3 > d_{\Gamma_1} + 2$ leads to trivial invariant. If α_2 and α_3 contribute to Y_2, then $l_1 \geq d_{\Gamma_1} + 1$. This leads to non-trivial invariant only if equality holds. That is, $\mu = \rho$ and $v_2 = 1$.

The remaining case is that $l_i = 2$, α_i contributes in Y_2 for all $i = 1, 2, 3$. We have $\mu = \rho$, $v_2 = 1$, $d = 3$, $d_{\Gamma_1} = 1$, $d_{\Gamma_2} = 4\rho + 2$. □

To summarize, notice that the weighted partitions associated to the relative invariants on the $Y_2 = \tilde{E}$ part are of the form $(\mu_1, \ldots, \mu_n) = (1, \ldots, 1)$ and $\deg \alpha_i = 2$ for all i, thus they are of the lowest order with fixed $|\mu|$. They can be reduced to absolute invariants readily.

For $\beta_2 = d_1\ell + d_2\gamma$, we see that $d_2 = \mu = \rho$ and so

$$d_{\Gamma_2} = 4d_2 + 2$$

is independent of d_1. Also d_2 is uniquely determined by the cohomology insertions. The presence of degenerate terms with degree β_2 for all large d_1 indicates the necessity of analytic continuations.

The same conclusion holds for $r \geq 3$ if we impose the nefness of K_X. We state the result in a slightly more general form:

Proposition B.13. *Let $\phi : Y \to X$ be the blow-up of X along a smooth center Z of dimension r and codimension $r' + 1$ with K_X nef and $r \leq r' + 1$. Then $C_\eta \neq 0$ only if $g_1 = 0$, $v_1 = \mu = \rho \neq 0$ and $\mu_1 \equiv 1$, $v_2 = 1$.*

The proof is entirely similar and we omit it.

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Legal disclaimer: For the bibliography, I have included mostly the related survey articles I have used. Please refer to the references therein or ask the experts in the school for further information. It is in no way my evaluation of the scholarly value... or lack-of!

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